Rather than giving the full generality we assume $p = n^{-\alpha}$ with $\frac{2}{3} > \alpha > 0$. The result is:

$$\Pr[G(n,p) \models A] = e^{-n^{4-6\alpha+o(1)}}$$

for $\frac{2}{3} > \alpha \geq \frac{2}{5}$ and

$$\Pr[G(n,p) \models A] = e^{-n^{2-\alpha+o(1)}}$$

for $\frac{2}{5} \geq \alpha > 0$.

The upper bound follows from the inequality

$$\Pr[G(n,p) \models A] \geq \max\left[(1-p^6)^{\binom{n}{4}}, (1-p)^{\binom{n}{2}}\right]$$

This is actually two inequalities. The first comes from the probability of G not containing a K_4 being at most the probability as if all the potential K_4 were independent. The second is the same bound on the probability that G doesn't contain a K_2 - i.e., that G has no edges. Calculation shows that the "turnover" point for the two inequalities occurs when $p = n^{-2/5+o(1)}$.

The upper bound follows from the Janson inequalities. For each four set α of vertices B_α is that that 4-set gives a K_4 and we want $\Pr[\wedge \overline{B_\alpha}]$. We have $\mu = \Theta(n^4p^6)$ and $-\ln M \sim \mu$ and (as shown in Lecture 1) $\Delta = \Theta(\mu\Delta^*)$ with $\Delta^* = \Theta(n^2p^5 + np^3)$. With $p = n^{-\alpha}$ and $\frac{2}{3} > \alpha > \frac{2}{5}$ we have $\Delta^* = o(1)$ so that

$$\Pr[\wedge \overline{B_\alpha}] \leq e^{-\mu(1+o(1))} = e^{-n^{4-6\alpha+o(1)}}$$

When $\frac{2}{5} > \alpha > 0$ then $\Delta^* = \Theta(n^2p^5)$ (somewhat surprisingly the np^3 never is significant in these calculations) and the extended Janson inequality gives

$$\Pr[\wedge \overline{B_\alpha} \leq e^{-\Theta(\mu^2/\Delta)} = e^{-\Theta(\mu/\Delta^*)} = e^{-n^{2-\alpha}}$$

The general result has been found by T. Luczak, A. Rucinski and S. Janson. Let H be any fixed graph and let A be the property of not containing a copy of H. For any subgraph H' of H the correlation inequality gives

$$\Pr[G(n,p) \models A] \leq e^{-E[X_{H'}]}$$

where $X_{H'}$ is the number of copies of H' in G. Now let $p = n^{-\alpha}$ where we restrict to those α for which p is past the threshold function for the appearance of H. Then

$$\Pr[G(n,p) \models A] = e^{n^{o(1)}} \min_{H'} e^{-E[X_{H'}]}$$

The reverse inequality was an open question for a full quarter century! Set $m = \lfloor n/\ln^2 n \rfloor$. For any set S of m vertices the restriction $G|_S$ has the distribution of $G(m, 1/2)$. Let $k = k(m) = k_0(m) - 4$ as above. Note

$$k \sim 2\log_2 m \sim 2\log_2 n$$

Then

$$\Pr[\alpha[G|_S] < k] < e^{-m^{2+o(1)}}$$

There are $\binom{n}{m} < 2^n = 2^{m^{1+o(1)}}$ such sets S. Hence

$$\Pr[\alpha[G|_S] < k \text{ for some } m\text{-set } S] < 2^{m^{1+o(1)}} e^{-m^{2+o(1)}} = o(1)$$

That is, almost always *every* m vertices contain a k-element independent set.

Now suppose G has this property. We pull out k-element independent sets and give each a distinct color until there are less than m vertices left. Then we give each point a distinct color. By this procedure

$$\begin{aligned}\chi(G) &\leq \lceil \tfrac{n-m}{k} \rceil + m \leq \tfrac{n}{k} + m \\ &= \tfrac{n}{2\log_2 n}(1+o(1)) + o(\tfrac{n}{\log_2 n}) \\ &= \tfrac{n}{2\log_2 n}(1+o(1))\end{aligned}$$

and this occurs for almost all G. □

3 Some Very Low Probabilities

Let A be the property that G does not contain K_4 and consider $\Pr[G(n,p) \models A]$ as p varies. (Results with K_4 replaced by an arbitrary H are discussed at the end of this section.) We know that $p = n^{-2/3}$ is a threshold function so that for $p \gg n^{-2/3}$ this probability is $o(1)$. Here we want to estimate that probability. Our estimates here will be quite rough, only up to a $o(1)$ additive factor in the hyperexponent, though with more care the bounds differ by "only" a constant factor in the exponent. If we were to consider all potential K_4 as giving mutually independent events then we would be led to the estimate $(1-p^6)^{\binom{n}{4}} = e^{-n^{4+o(1)}p^6}$. For p appropriately small this turns out to be correct. But for, say, $p = \frac{1}{2}$ it would give the estimate $e^{-n^{4+o(1)}}$. This must, however, be way off the mark since with probability $2^{-\binom{n}{2}} = e^{-n^{2+o(1)}}$ the graph G could be empty and hence trivially satisfy A.

Then $n = \sqrt{2}^{k(1+o(1))}$ so for $k \sim k_0$,

$$f(k+1)/f(k) = \frac{n}{k}2^{-k}(1+o(1)) = n^{-1+o(1)}$$

Set

$$k = k(n) = k_0(n) - 4$$

so that

$$f(k) > n^{3+o(1)}$$

Now we use the Generalized Janson Inequality to estimate $\Pr[\omega(G) < k]$. Here $\mu = f(k)$. (Note that Janson's Inequality gives a lower bound of $2^{-f(k)} = 2^{-n^{3+o(1)}}$ to this probability but this is way off the mark since with probability $2^{-\binom{n}{2}}$ the random G is empty!) The value Δ was examined in Lecture 2 and we showed

$$\frac{\Delta}{\mu^2} = \frac{\Delta^*}{\mu} = \sum_{i=2}^{k-1} g(i)$$

There $g(2) \sim k^4/n^2$ and $g(k-1) \sim 2kn2^{-k}/\mu$ were the dominating terms. In our instance $\mu > n^{3+o(1)}$ and $2^{-k} = n^{-2+o(1)}$ so $g(2)$ dominates and

$$\Delta \sim \frac{\mu^2 k^4}{n^2}$$

Hence we bound the *clique* number probability

$$\Pr[\omega(G) < k] < e^{-\mu^2(1+o(1))/2\Delta} = e^{-(n^2/k^4)(1+o(1))} = e^{-n^{2+o(1)}}$$

as $k = \Theta(\ln n)$. (The possibility that G is empty gives a lower bound so that we may say the probability is $e^{-n^{2+o(1)}}$, though a $o(1)$ in the hyperexponent leaves lots of room.)
Theorem 2.1. (Bollobás [1988]) Almost always

$$\chi(G)) \sim \frac{n}{2\log_2 n}$$

Proof. The argument that

$$\chi(G) \geq \frac{n}{\alpha(G)} \geq \frac{n}{2\log_2 n}(1+o(1))$$

almost always was given in Lecture 2.

As λ ranges from $-\infty$ to $+\infty$, $e^{-e^{\lambda}}$ ranges from 1 to 0. As $n_0(k+1) \sim \sqrt{2}n_0(k)$ the ranges will not "overlap" for different k. More precisely, let K be arbitrarily large and set

$$I_k = [n_0(k)[1 - \frac{K}{k}], n_0(k)[1 + \frac{K}{k}]]$$

For $k \geq k_0(K)$, $I_{k-1} \cap I_k = \emptyset$. Suppose $n \geq n_0(k_0(K))$. If n lies between the intervals (which occurs for "most" n), which we denote by $I_k < n < I_{k+1}$, then

$$\Pr[\omega(G(n,p)) < k] \leq e^{-e^{K}} + o(1),$$

nearly zero, and

$$\Pr[\omega(G(n,p)) < k+1] \geq e^{-e^{-K}} + o(1),$$

nearly one, so that

$$\Pr[\omega(G(n,p)) = k] \geq e^{-e^{-K}} - e^{-e^{K}} + o(1),$$

nearly one. When $n \in I_k$ we still have $I_{k-1} < n < I_{k+1}$ so that

$$\Pr[\omega(G(n,p)) = k \textit{ or } k-1] \geq e^{-e^{-K}} - e^{-e^{K}} + o(1),$$

nearly one. As K may be made arbitrarily large this yields the celebrated two point concentration theorem on clique number given as Corollary 2.1.2. Note, however, that for most n the concentration of $\omega(G(n,1/2))$ is actually on a single value!

2 Chromatic Number

Again fix $p = 1/2$ (there are similar results for other p) and let $G \sim G(n, \frac{1}{2})$. We shall find bounds on the chromatic number $\chi(G)$. The original proof of Bollobás used martingales and will be discussed later. Set

$$f(k) = \binom{n}{k} 2^{-\binom{k}{2}}$$

Let $k_0 = k_0(n)$ be that value for which

$$f(k_0 - 1) > 1 > f(k_0)$$

Lecture 4: The Chromatic Number Resolved!

The centerpiece of this lecture is the result of Béla Bollobás that, with $G \sim G(n, \frac{1}{2})$, $\chi(G) \sim n/(2\log_2 n)$ almost surely.

1 Clique Number Revisited

In this section we fix $p = 1/2$, (other values yield similar results), let $G \sim G(n,p)$ and consider the clique number $\omega(G)$. For a fixed $c > 0$ let $n, k \to \infty$ so that

$$\binom{n}{k} 2^{-\binom{k}{2}} \to c$$

As a first approximation

$$n \sim \frac{k}{e\sqrt{2}} \sqrt{2}^k$$

and

$$k \sim \frac{2\ln n}{\ln 2}$$

Here $\mu \to c$ so $M \to e^{-c}$. The Δ term was examined earlier. For this k, $\Delta = o(E[X]^2)$ and so $\Delta = o(1)$. Therefore

$$\lim_{n,k\to\infty} \Pr[\omega(G(n,p)) < k] = e^{-c}$$

Being more careful, let $n_0(k)$ be the minimum n for which

$$\binom{n}{k} 2^{-\binom{k}{2}} \geq 1.$$

Observe that for this n the left hand side is $1 + o(1)$. Note that $\binom{n}{k}$ grows, in n, like n^k. For any $\lambda \in (-\infty, +\infty)$ if

$$n = n_0(k)[1 + \frac{\lambda + o(1)}{k}]$$

then

$$\binom{n}{k} 2^{-\binom{k}{2}} = [1 + \frac{\lambda + o(1)}{k}]^k = e^{\lambda} + o(1)$$

and so

$$\Pr[\omega(G(n,p)) < k] = e^{-e^{\lambda}} + o(1)$$