THE PROBABILISTIC METHOD

Noga Alon

Department of Mathematics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
Tel Aviv, Israel

Joel H. Spencer

Courant Institute of Mathematical Sciences
New York University
New York, USA

With an Open Problems Appendix by **Paul Erdős**

To Nurit and Maryann

Preface

The Probabilistic Method has recently been developed intensively and became one of the most powerful and widely used tools applied in Combinatorics. One of the major reasons for this rapid development is the important role of randomness in Theoretical Computer Science, a field which is recently the source of many intriguing combinatorial problems.

The interplay between Discrete Mathemtics and Computer Science suggests an algorithmic point of view in the study of the Probabilistic Method in Combinatorics and this is the approach we tried to adopt in this book. The manuscript thus includes a discussion of algorithmic techniques together with a study of the classical method as well as the modern tools applied in it. The first part of the book contains a description of the tools applied in probabilistic arguments, including the basic techniques that use expectation and variance, as well as the more recent applications of Martingales and Correlation Inequalities. The second part includes a study of various topics in which probabilistic techniques have been successful. This part contains chapters on discrepancy and random graphs, as well as on several areas in Theoretical Computer Science; Circuit Complexity, Computational Geometry, and Derandomization of randomized algorithms. Scattered between the chapters are gems described under the heading "The Probabilistic Lens". These are elegant proofs that are not necessarily related to the chapters after which they appear and can be usually read separately.

The basic Probabilistic Method can be described as follows: in order to prove the existence of a combinatorial structure with certain properties, we construct an appropriate probability space and show that a randomly chosen element in this space has the desired properties with positive probability. This method has been initiated by Paul Erdős, who contributed so much to its development over the last forty five years, that it seems appropriate to call it "The Erdős Method". His contribution cannot be measured only by his numerous deep results to the subject, but also by his many intriguing problems and conjectures that stimulated a big portion of the research in the area. It is therefore very pleasing that we can close this book with an Open Problems Appendix by Paul Erdős, the founder of the Probabilistic Method.

It seems impossible to write an encyclopedic book on the Probabilitic Method; too many recent interesting results apply probabilistic arguments, and we do not even try to mention all of them. Our emphasis is on methodology, and we thus try to describe the ideas, and not always to give the best possible results if these are too technical to allow a clear presentation. Many of the results are asymptotic, and we use the standard asymptotic notation: for two functions f and g, we write $f = O(g)$ if $f \leq c_1 g + c_2$ for all possible values of the variables of the two functions, where c_1, c_2 are absolute constants. We write $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. If the limit of the ratio f/g tends to zero as the variables of the functions tend to infinity we write $f = o(g)$. Finally, $f \sim g$ denotes that $f = (1 + o(1))g$, i.e., that f/g tends to 1 when the variables tend to infinity.

It is a special pleasure to thank our wives, Nurit and Maryann. Their patience, understanding and encouragment have been a key-ingredient in the success of this enterprise.

November 1990, Tel Aviv and New York.

TABLE OF CONTENTS

PART I: METHODS