Chapter 0: An Infinity of Primes

Truth is on a curve whose asymptote our spirit follows eternally.
Léo Errera

We begin with one of the greatest theorems in mathematics.

Theorem 0.1 *There is an Infinite Number of Primes*

Our proof is not that of Euclid and not better than the proof of Euclid, but it illustrates the theme of this work: looking at the mathematical world through an asymptotic lens.

We begin as Euclid did. Assume Theorem 0.1 is false. Let $p_1, \ldots, p_r$ be a listing of all of the primes. For any nonnegative integer s the Unique Factorization Theorem states that there is a unique way to express

$$s = p_1^{\alpha_1} p_2^{\alpha_2} \cdot p_r^{\alpha_r} \tag{1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are nonnegative integers. We turn this into an *encoding* of the nonnegative integers by creating a map Ψ

$$\Psi(s) = (\alpha_1, \ldots, \alpha_r) \tag{2}$$

Let $n \geq 2$ be arbitrary, though in application below it shall be large. The integers s, $1 \leq s \leq n$ are each mapped by Ψ to a vector of length r. How many possibilities are there for the values $\Psi(s)$? We give an *upper bound*. For each $1 \leq i \leq r$ the value α_i must satisfy

$$p_i^{\alpha_i} \leq s \leq n \tag{3}$$

Thus

$$\alpha_i \leq \log_{p_i} n \leq \log_2 n \tag{4}$$

As α_i is a nonnegative integer there are at most $1 + \log_2 n$ possibilities for it. With $n \geq 2$ the number of possibilities is at most $2 \log_2 n$. (These kind of gross upper bounds will appear quite often and a large part of the *art* of asymptopia is knowing when to use them and when not to use them.) Thus the number of possible values of $\Psi(s) = (\alpha_1, \ldots, \alpha_r)$ is at most $2^r (\log_2 n)^r$. The vectors $(\alpha_1, \ldots, \alpha_r)$ uniquely determine s by Equation 1. We have n *different* values $\Psi(s)$. We deduce that

$$2^r (\log_2 n)^r \geq n \tag{5}$$

The above is all true for any $n \geq 2$. But now we apply an asymptotic lens and consider (5) asymptotically in n. The left hand side is a constant times a fixed power of the logarithm function. We know (more on this later in the book) that any fixed power of the $\ln n$ grows slower that any fixed positive power of n, so slower than n itself. This means that for n sufficiently large, (5) must fail! We have achieved our *reductio ad absurdum*, the assumption that the number of primes is finite must be false and Theorem 0.1 is true.

`Remark:` We do not need the full power of the Unique Factorization Theorem. It suffices to know that every s has *some* representation Equation 1 as the product of primes to powers. Then for each of the $1 \leq s \leq n$ select arbitrarily one such representation as $\Psi(s)$. One still has n distinct $\Psi(s)$ and at most $2^r(\log_2 n)^r$ possible vectors $(\alpha_1, \ldots, \alpha_r)$.

We have worked out this argument in some detail. For those comfortable with asymptotics it would go, informally, something like this: There are n values $\Psi(s)$, $1 \leq s \leq n$ and logarithmically many values for each coordinate α_i and therefore *polylog* many vectors, but polylog grows slower than linear.

We now use this same approach to prove a much stronger result:

Theorem 0.2 *The summation of the reciprocals of the primes diverges.*

`Proof:` Again, assume not. So

$$\sum_p \frac{1}{p} = C \tag{6}$$

for some constant C. (We shall use $\sum_p$ to indicate the sum over all primes p.) Label the primes $p_1, p_2, \ldots$. As Equation 6 is a convergent sum of positive terms, at some point it reaches $C - \frac{1}{2}$. That is, there exists r such that

$$\sum_{i>r} \frac{1}{p_i} < \frac{1}{2} \tag{7}$$

Call the primes $p_1, \ldots, p_r$ small and the other primes large. Call an integer s *rare* if all of its prime factors are small, otherwise call s *ordinary*. Again consider the s, $1 \leq s \leq n$. The rare integers have a factorization (1) and so, as with Theorem 0.1 their number is at most most $2^r(\log_2 n)^r$, polylog to the cognoscenti.

What about the ordinary s? For each ordinary s there is some (perhaps several) large prime p dividing it. For a given prime p, the number of elements in $1 \leq s \leq n$ divisible by it is $\lfloor \frac{n}{p} \rfloor$, which is at most $\frac{n}{p}$. Thus, the total number of ordinary s is at most $\sum \frac{n}{p}$ where p now ranges over the large primes. From (7), this is less than $\frac{n}{2}$. Of the n values of s, less than $\frac{n}{2}$ are ordinary, so at least $\frac{n}{2}$ are rare. Thus

$$2^r(\log_2 n)^r \geq \frac{n}{2}. \tag{8}$$

As with (5), for n sufficiently large (5) must fail! We have achieved our *reductio ad absurdum*, the assumption that the sum of the reciprocals of the primes is finite must be false and Theorem 0.2 is true.

`Remark:` There was no need to cut large and small primes precisely via (7). The same argument works if the sum of the reciprocals of the large primes is less than one.