Honors Algebra II
Assignment 9 Solutions

1. Let $F \subset L \subset K$ be fields in $C$. Assume that $K : F$ is normal and that $L : F$ is normal. Let $\tau \in \Gamma[K : F]$ and $\sigma \in \Gamma[K : L]$. Let $l \in L$

(a) Using a result shown in class argue that $\tau(l) \in L$.

Solution: We showed that when $L : F$ was normal that any isomorphism $\gamma : L \to L'$ over $F$ had $L' = L$. The restriction of $\tau$ to $L$ is such an isomorphism so $L' = L$.

(b) Show that $(\tau \sigma \tau^{-1})(l) = l$.

Solution: Set $\tau(l) = l'$ so $l' \in L$. So $\sigma(l') = l'$. So $\tau^{-1}(l') = l$. That is $(\tau \sigma \tau^{-1})(l) = l$.

(c) From the above show that $\Gamma[K : L]$ is a normal subgroup (get out those Algebra I notes!) of $\Gamma[K : F]$. (Assume its already been shown that it is a subgroup. You only need show the normal part.)

Solution: We’ve shown that if $\tau \in \Gamma[K : F]$ and $\sigma \in \Gamma[K : L]$ then $\tau \sigma \tau^{-1}$ fixes all elements of $L$ so $\tau \sigma \tau^{-1} \in \Gamma[K : L]$. That is what we need to show that you have a normal subgroup.

(d) In the case $F = Q$, $L = Q(\omega)$, $K = F(\alpha, \omega)$ (with $\alpha = 2^{1/3}, \omega = e^{2\pi i/3}$ as in our standard example) give the groups $\Gamma[K : L]$ and $\Gamma[K : F]$ explicitly in terms of permutations of $\alpha, \beta = \alpha \omega, \gamma = \alpha \omega^2$.

Solution: $\Gamma[K : F]$, as done in class, is all six permutations of $\alpha, \beta, \gamma$. Now fix $L$, so $\tau(\omega) = \omega$. There are three cases. Either $\tau(\alpha) = \alpha$ so $\tau = e$. Or $\tau(\alpha) = \beta = \alpha \omega$. Then $\tau(\beta) = \tau(\alpha \omega) = \alpha \omega \omega = \gamma$ and similarly $\tau(\gamma) = \alpha$. So $\tau$ sends $\alpha, \beta, \gamma$ into $\beta, \gamma, \alpha$ respectively. Similarly if $\tau(\alpha) = \gamma$, $\tau$ sends $\alpha, \beta, \gamma$ into $\gamma, \alpha, \beta$ respectively.

2. Let $F \subset K$ be fields in $C$ with $[K : F] = 2$. Prove that $K : F$ is a normal extension.

Solution: Let $K : F$ be an extension of degree 2. Then $K = F(\alpha)$ for some $\alpha$, where the minimal polynomial of $\alpha$ in $K[x]$ has degree 2. Let $m(x) \in K[x]$ be the minimal polynomial of $\alpha$, so

$$m(x) = x^2 + bx + c$$

for some $b, c \in K$. (We may assume $m(x)$ is monic to make the calculations easier.) We will show $K : F$ is normal by showing that $K$
is a splitting field for the polynomial $m(x)$ over $F$. By the quadratic formula, the roots of $m(x)$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

As $m(x)$ is irreducible the roots are distinct. Suppose

$$\alpha = \frac{-b + \sqrt{b^2 - 4c}}{2},$$

(the case where the square root is subtracted is similar) then

$$\beta = \frac{-b - \sqrt{b^2 - 4c}}{2} = \frac{b - \sqrt{b^2 - 4c} - 2b}{2} = -\alpha - b.$$

Therefore, $\alpha, \beta \in F(\alpha) = K$. (Another approach: Letting $\alpha, \beta$ be the roots, $\alpha + \beta = -b$ immediately from the connection between the roots of a polynomial and its coefficients.)

3. Let $K_1, K_2$ be normal extensions of $Q$. Let $M$ denote the minimal field containing $K_1 \cup K_2$. Prove that $M$ is a normal extension of $Q$. [One approach: Write $K_1 = Q(\alpha_1, \ldots, \alpha_r)$ where the $\alpha_i$ are all the roots of some $p(x) \in Q[x]$ and do similarly for $K_2$.]

**Solution:** Write $K_1$ as above and similarly $K_2 = Q(\beta_1, \ldots, \beta_s)$ where the $\beta_j$ are all the roots of some $q(x) \in Q[x]$. Then $M = Q(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s)$ which are all the roots of $p(x)q(x)$ which is in $Q[x]$. (Reminder: To be a splitting field you don’t need to have an irreducible polynomial!)

4. Let $\alpha$ be a root of $f(x) = x^3 + x^2 - 2x - 1 \in Q[x]$.

   (a) Show $f(x)$ is irreducible over $Q$.

   **Solution:** It is a cubic and by Gauss’s Theorem if it factors it would factor into $Z[x]$ so it would a factor $x - a$ so it would have an integer root $a$. Then $a$ would have to divide $-1$ so it would be either $+1$ or $-1$, and neither of them work.

   (b) Find $[Q(\alpha) : Q]$.

   **Solution:** $f(x)$ is irreducible over $Q$ and $\alpha$ is a root of $f(x)$, so $f(x)$ is the minimal polynomial for $\alpha$ over $Q$. Hence $[Q(\alpha) : Q] = \deg(f) = 3$.

   (c) Set $\beta = -1/(\alpha + 1)$. Find $\beta$ in the form $a + b\alpha + c\alpha^2$.

   **Solution:** First lets find $1/(1 + \alpha)$: $(\alpha + 1)(a + b\alpha + c\alpha^2) =$
\[ a + b\alpha + c\alpha^2 + a\alpha + b\alpha^2 + c(-\alpha^2 + 2\alpha + 1) = 1 \]
gives the equations
\[ a + c = 1, \ b + a + 2c = 0, \ c + b - c = 0 \]
so \( b = 0, \ a = 2, \ c = -1. \)
Thus \( \beta \) is the negative of that: \( \beta = \alpha^2 - 2. \)

(d) Show that \( f(\beta) = 0. \) (Bit of grunt work here!)

Solution: We need show
\[ -\frac{1}{(1 + \alpha)^3} + \frac{1}{(1 + \alpha)^2} - 2\frac{-1}{1 + \alpha} - 1 = 0 \]

Multiplying out by \((1 + \alpha)^3\) we get that a polynomial which is zero. Or, we need show
\[ (\alpha^2 - 2)^3 + (\alpha^2 - 2)^2 - 2(\alpha^2 - 2) - 1 = 0 \]
One makes a list of \( \alpha^3, \alpha^4, \alpha^5, \alpha^6 \) and everything works out.

(e) Find \( \gamma \in Q(\alpha), \gamma, \beta, \alpha \) distinct, with \( f(\gamma) = 0. \) (Idea: If \( f(x) = (x - \alpha)(x - \beta)(x - \gamma) \) then \( \alpha + \beta + \gamma \) is determined.)

Solution: \( \alpha + \beta + \gamma = -1 \) so \( \gamma = -1 - \alpha - \beta. \)

(f) Deduce that \( Q(\alpha) : Q \) is normal.

Solution: It is the splitting field of \( f(x). \)

(g) List all of the \( Q \)-automorphisms of \( Q(\alpha). \) What familiar group is \( \Gamma(Q(\alpha) : Q) \) isomorphic to?

Solution: When \( [Q(\gamma) : Q] = r \) and \( Q(\gamma) \) is normal over \( Q \) then the Galois group has precisely \( r \) elements. But there is precisely one group on three elements. So \( \Gamma(Q(\alpha) : Q) \) must be isomorphic to \( Z_3. \) One is the identity. Then there is a \( \sigma \) with \( \sigma(\alpha) = \beta. \) We can’t have \( \sigma(\gamma) = \gamma \) as that would make \( \sigma \) the identity (as \( Q(\alpha) = Q(\gamma) \)) so \( \sigma(\gamma) = \alpha \) and \( \sigma(\beta) = \gamma. \) The final automorphims sends \( \alpha, \beta, \gamma \) to \( \alpha, \gamma, \beta \) respectively.

(h) Let \( K \) be the fixed field of \( \Gamma(Q(\alpha) : Q). \) Prove \( K = Q. \)

Solution: This is the Galois Correspondence Theorem – the entire Galois Group corresponds to the ground field.

5. As in last week’s assignment set \( \alpha = 2^{1/4}, \beta = i\alpha, \gamma = -\alpha, \delta = -i\alpha. \)
Set \( p(x) = x^4 - 2. \) Set \( K = Q(\alpha, \beta, \gamma, \delta). \) Set \( L = Q(i). \) Also, set \( M = Q(\alpha) \) and \( N = Q(\sqrt{2}). \)

(a) Give the factorization of \( p(x) \) into irreducible factors in \( Q[x]. \)

Solution: It is irreducible by the Eisenstein Criterion.

(b) Give the factorization of \( p(x) \) into irreducible factors in \( K[x]. \)

Solution: It completely factors \( p(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta). \)
(c) Give the factorization of $p(x)$ into irreducible factors in $M[x]$.

**Solution:** Well, $(x - \alpha)$ is a factor. But as $\gamma = -\alpha \in M$ so is $(x - \gamma) = (x + \alpha)$. Taking those out $p(x) = (x - \alpha)(x + \alpha)(x^2 + \sqrt{2})$. Here $\sqrt{2} = \alpha^2 \in M$. Also, $x^2 + \sqrt{2}$ is irreducible in $M[x]$ as its two roots, $\gamma, \delta$, are not in $M$ as they are complex numbers.

(d) Give the factorization of $p(x)$ into irreducible factors in $N[x]$.

**Solution:** $p(x) = (x^2 - \sqrt{2})(x^2 + \sqrt{2})$. As $[Q(\alpha) : Q] = 4$, $\alpha \notin N$. Thus $(x - \alpha)$ is not a factor in $N[x]$. Similarly $x - \beta$, $x - \gamma$, $x - \delta$ are not factors. Thus the quadratics above are irreducible in $N[x]$. 