Algebra II
Assignment 5
Solutions

1. Let $\alpha = a + bi \in C$. Let $\beta = c + di \in C$ be such that $\beta^2 = \alpha$. (That is, $\beta$ is one of the two square roots of $\alpha$.) Let $K$ be a field with $Q \subseteq K$ and $a, b \in K$. Find an explicit tower (you find the $r$!)

$$K = K_0 \subset K_1 \subset \ldots \subset K_r = L$$

with all $[K_{i+1} : K] = 2$ and $c, d \in L$. (This is a bit tricky. Its helpful to think in terms of polar coordinates and first find the “$r$”-value for $\beta$.

Solution: Set $R = \sqrt{a^2 + b^2}$ so that in polar coordinates $\alpha = (R, \theta)$. and so $\beta = (\sqrt{R}, \theta/2)$. Squaring $\beta$ gives the equations

$$c^2 - d^2 = a$$

$$2cd = b$$

Look tough. But we also know

$$\sqrt{c^2 + d^2} = \sqrt{R}$$

so that

$$c^2 + d^2 = R$$

Adding and subtracting the first equation gives

$$c^2 = \frac{R + a}{2}$$

$$d^2 = \frac{R - a}{2}$$

which give $c, d$. (Of the four possibilities for $c, d$, taking the two square-roots, exactly two pair work giving $2cd$ correctly.) So to get the tower of fields we take $K_1 = K(R)$ and then $K_2 = K_1(\sqrt{(R + a)/2})$. As $d = b/2c$ we have $d$ for free.

Another Approach: Substituting $d = \frac{b}{2c}$ into the equation $c^2 - d^2 = a$ gives $c^2 - \frac{b^2}{4c^2} = a$ or $4c^4 - b^2 = ac^2$. This looks like a quartic in $c$ but because we only have even powers we can reduce it by setting $e = c^2$, solving the now quadratic for $e$, and then getting $c = \sqrt{e}$. 
2. Let $\epsilon = e^{2\pi i/5}$. Set $K = Q(\epsilon)$.

(a) Find $[K : Q]$ and a basis for $K$ over $Q$.

Solution: As $[(x^5 - 1)/(x - 1)]$ is irreducible, $[K : Q] = 4$ with basis $1, \epsilon, \epsilon^2, \epsilon^3$.

(b) Set $\gamma = \epsilon + \epsilon^4$. Show that $[Q(\gamma) : Q] = 2$ by finding an explicit quadratic equation, with coefficients in $Q$, satisfied by $\gamma$. (Note: Since you have the basis this is a linear algebra problem: finding a dependence between $1, \gamma, \gamma^2$.)

Solution: We have $\gamma^2 = \epsilon^2 + 2\epsilon^5 + \epsilon^8$ but as $\epsilon^5 = 1$ we have $\gamma^2 = 2 + \epsilon^2 + \epsilon^3$. Also $\gamma = -1 - \epsilon^2 - \epsilon^3$ as $1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 = 0$. Now we have $1, \gamma, \gamma^2$ in terms of the four basis elements. Normally three vectors in 4-space aren’t dependent but these are. We either eyeball or we use undetermined coefficient: $a + b\gamma + c\gamma^2 = 0$. Then we get four equations: $a - b + 2c = 0$ (the constant), $0 = 0$ (as there is no coefficient of $\epsilon$), $b - c = 0$ (for $\epsilon^2$) and $b - c = 0$ (for $\epsilon^3$) which means $-a = b = c$. Thus $a = -1, b = c = 1$ is a solution, $-1 + \gamma + \gamma^2 = 0$.

(c) Use the quadratic formula to solve $\gamma$ explicitly. Find an explicit $d \in Z$ with $Q(\gamma) = Q(\sqrt{d})$.

Solution: $\gamma = [-1 \pm \sqrt{5}]/2$. From the geometry we see $\gamma$ is positive so it is actually $\gamma = [-1 + \sqrt{5}]/2$. Thus $Q(\gamma) = Q(\sqrt{5})$.

(d) Show $[K : Q(\gamma)] = 2$. (This is immediate if you see it.)

Solution: The product theorem as $[K : Q] = [K : Q(\gamma)] \cdot [Q(\gamma) : Q]$ and we know $[K : Q] = 4$ and $[Q(\gamma) : Q] = 2$.

(e) As $\epsilon \in K$, find an explicit quadratic equation, with coefficients in $Q(\gamma)$, satisfied by $\epsilon$.

Solution: One nice way is to write $\gamma = \epsilon + \epsilon^4$ and multiply by $\epsilon$ to give $\epsilon\gamma = \epsilon^2 + \epsilon^5 = \epsilon^2 + 1$ so we have the quadratic $\epsilon^2 - \epsilon\gamma + 1 = 0$.

(f) Use the quadratic formula to solve $\epsilon$ explicitly. (Some of the terms will be square roots of non-real numbers, but let’s allow that. The object is to write $\epsilon$ in terms of usual field expressions and square roots.)

Solution: By the quadratic formula

$$\epsilon = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$
As $\gamma \sim 0.6\ldots$, the square root is imaginary and can be written $i\sqrt{1-\gamma^2}$.

(g) Write $\epsilon = a + bi$. Find $a, b$ in terms of usual field expressions and square roots, but not involving complex numbers. 

Solution: From above we find 

$$a = \frac{\gamma}{2} = \frac{-1 + \sqrt{5}}{4}$$

and 

$$b = \frac{\sqrt{1-\gamma^2}}{2} = \frac{\sqrt{\frac{\sqrt{5}-1}{2}}}{2}$$

3. Let $f(x) = x^3 + ax^2 + cx + d \in Q[x]$ be an irreducible cubic with one real root $\alpha$ and two nonreal roots $\beta, \gamma$.

(a) Argue that $\gamma = \overline{\beta}$. (Note: Here, and often, we let $\overline{\kappa}$ denote the complex conjugate of $\kappa$.)

Solution: $\gamma, \beta$ satisfy a quadratic $f(x)/(x - \alpha)$ which has real roots, call it $x^2 + ax + b = 0$. We don’t have $a^2 - 4b \geq 0$ as then $\gamma, \beta$ would be real so we have $a^2 - 4b < 0$ and the quadratic formula gives two roots which are complex conjugates.

(b) Argue that $\gamma \not\in Q(\alpha)$. 

Solution: All elements in $Q(\alpha)$ are real.

(c) Argue that $[Q(\alpha, \gamma) : Q(\alpha)] = 2$. 

Solution: From above $\gamma$ satisfies a quadratic $f(x)/(x - \alpha)$ in $Q(\alpha)[x]$ but as $\gamma \not\in Q(\alpha)$ it can’t satisfy a linear polynomial, so its minimal polynomial over $Q(\alpha)$ is of degree 2 and so that is $Q(\alpha, \gamma) : Q(\alpha)$.

(d) Show that $\beta \in Q(\alpha, \gamma)$ and that $\alpha \in Q(\beta, \gamma)$. 

Solution: As $f(x)$ has roots $\alpha, \beta, \gamma$ we can write $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$. Equating coefficients gives 

$$-\alpha - \beta - \gamma = a$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = b$$

$$-\alpha\beta\gamma = c$$

Taking, say, the first, $\beta = -\alpha - \gamma - a \in Q(\alpha, \gamma)$ and similarly $\alpha = -\beta - \gamma - a \in Q(\beta, \gamma)$. 

(e) Argue that \([Q(\alpha, \beta, \gamma) : Q] = 6\).

Solution: We have a tower \(Q \subset Q(\alpha) \subset Q(\alpha, \gamma) = Q(\alpha, \beta, \gamma)\)
and \([Q(\alpha) : Q] = 3\) and \([Q(\alpha, \gamma) : Q(\alpha)] = 2\) so be the tower theorem \([Q(\alpha, \beta, \gamma) : Q] = 6\).

(f) Argue that \(\gamma \notin Q(\beta)\). (Idea: If \(\gamma \in Q(\beta)\) then show that \(Q(\alpha, \beta, \gamma) = Q(\beta)\) and get a contradiction.)

Solution: If it were \(Q(\beta, \gamma) = Q(\beta)\) so \(Q(\alpha, \beta, \gamma) = Q(\beta, \gamma) = Q(\beta)\). But \([Q(\alpha, \beta, \gamma) : Q] = 6\) and \([Q(\beta) : Q] = 3\).

Remark: By the last part, \(Q(\beta)\) is a field which is not closed under complex conjugation.