1. In this problem, assume (important!) that $G$ is an Abelian. Set $H = \{g \in G : g^5 = e\}$. (Warning: Expressions such as $x^{1/5}$ are not well defined. Do not use them!)

(a) Show $H$ is a subgroup of $G$. Point out where the assumption that $G$ was Abelian was used.

Solution. As usual, three parts.

Identity: As $e^5 = e$, $e \in H$.

Product: If $x, y \in H$ then $x^5 = y^5 = e$ so that $(xy)^5 = x^5y^5 = ee = e$ so that $xy \in H$ – where we need that the group is Abelian to say that

$$(xy)^5 = xyxyxyxy = xxxxyyyy = x^5y^5$$

Inverse: If $xin H$ then $x^5 = e$ so that $(x^{-1})^5 = (x^5)^{-1} = e^{-1} = e$ so $x^{-1} \in H$.

(b) Show $H$ is a normal subgroup of $G$.

$G$ is Abelian so that all subgroups of $G$ are Normal!

(c) Assume further that $G$ is finite and that $H = \{e\}$. Show that the map $\phi : G \to G$ given by $\phi(g) = g^5$ is an automorphism.

(Definition: An automorphism is an isomorphism from a group to itself.)

Proof: $\phi$ is a homomorphism as

$$\phi(xy) = (xy)^5 = x^5y^5 = \phi(x)\phi(y)$$

for all $x, y \in G$

$\phi$ is injective as the kernel

$$K_\phi = \{g \in G : \phi(g) = e\} = H = \{e\}$$

by assumption, and we proved in class that $\phi$ is injective if and only if the kernel is $\{e\}$. Finally, since $G$ is finite, an injective map from $G$ to itself must be surjective. This is the pigeonhole principle. Say $G$ has $m$ elements. We have a map $\phi$ from $m$ elements (pigeons) to $m$ elements (pigeonholes) and no two pigeons go into the same pigeonhole (injectivity) so all pigeonholes are filled (surjectivity). BTW – it goes the other way – surjectivity would imply injectivity. But the set MUST be finite!
2. Let $G$ be any group and $g$ a fixed element of $G$. Define $\phi : G \to G$ by $\phi(x) = g^{-1}xg$.

(a) Show that $\phi$ is a homomorphism.
Solution:
\[
\phi(xy) = gxg^{-1}gyg^{-1} = g(xy)g^{-1} = \phi(x)\phi(y)
\]

(b) Show that $\phi$ is an injection. To do this you have to show that the equation $\phi(x) = e$ has only the solution $x = e$.
Solution: If $g^{-1}xg = e$ then premultiply by $g$, postmultiply by $g^{-1}$ so $x = gg^{-1} = e$.

(c) Show that $\phi$ is a surjection. To do this you have to show, given any $y \in G$, that the equation $\phi(x) = y$ has a solution. [With these three you may conclude that $\phi$ is an automorphism as defined above.]
Solution: Set $x = ggy^{-1}$.

(d) Find an isomorphism $\phi : (\mathbb{Z}_{12}, +) \to (\mathbb{Z}_{13}^*, \cdot)$. (Idea: $\phi(1)$ determines $\phi$, try various $\phi(1)$ until one works.)
Solution: $\phi(1) = 2$ works (others do as well) giving
\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7
\end{array}
\]

(e) Let $G$ be the positive reals under multiplication and let $H$ be numbers $2^i$ where $i \in \mathbb{Z}$.

i. Show $H$ is a subgroup of $G$.
Solution: $1 = 2^0 \in H$. If $x, y \in H$ then $x = 2^i, y = 2^j$ so $xy = 2^{i+j} \in H$. If $x \in H$ then $x = 2^i$ so $x^{-1} = 2^{-i} \in H$.

ii. Show $H$ is a Normal subgroup of $G$.
Solution: $G$ is Abelian so all subgroups are Normal.

iii. Give a natural representation of the factor group $G/H$. By this I mean that each element should be uniquely describable as $\overline{a}$ where $a$ ranges over some natural set. (So, for example, you couldn’t have 3.1 and 12.4 as $\overline{3.1} = \overline{12.4}$.) Have the identity represented as $\overline{1}$.
Solution: Restrict $a$ to $1 \leq a < 2$. The $\overline{a}$ are distinct as if $1 \leq a < b < 2$ then $1 < b/a < 2$ so $b/a \notin H$ so $\overline{a} \neq \overline{b}$. But given any $x \in G$ we can write $x = 2^ia$ with $1 \leq a < 2$ so $\overline{x} = \overline{a}$.
iv. Find all elements $\overline{a} \in G/H$ whose cube (in $G/H$) is the identity.

Solution: There are three: $\overline{1}, \overline{\alpha}, \overline{\beta}$, where $\alpha = 2^{1/3}$ (so $\alpha^3 = \overline{2} = \overline{1}$) and $\beta = 2^{2/3}$ (so $\beta^3 = \overline{4} = \overline{1}$).

(f) Just for fun: What's purple and commutes?

Standard Solution: An Abelian grape!

Alternate Solution: NYU student living in Queens

(g) Some questions about the order of an element.

i. Let $g \in G$ with $o(g) = 100$. For what $i$ is $g^i = e$?

Solution: Exactly the multiples of 100.

ii. Let $g \in G$ with $g^{100} = e$. What are the possible values of $o(g)$?

Solution: Exactly the divisors 1, 2, 4, 5, 10, 20, 25, 50, 100 of 100.

iii. Let $\phi : G \to H$ be a homomorphism and let $g \in G$ and set $h = \phi(g)$. Suppose $o(h) = 100$. Assume $g$ has finite order. What are the possible values of $o(g)$?

Solution: If $g$ has order $n$ then $g^n = e$ so $e = \phi(g^n) = h^n$. Thus $n$ must be a multiple of 100.