Honors Algebra II
Assignment 10 Solutions

1. Let \( F \subset L \subset K \) be fields in \( \mathbb{C} \). Assume that \( K : F \) is normal and that \( L : F \) is normal. Let \( \tau \in \Gamma[K:F] \) and \( \sigma \in \Gamma[K:L] \). Let \( l \in L \)

   (a) Using a result shown in class argue that \( \tau(l) \in L \).
   \textbf{Solution:} We showed that when \( L : F \) was normal that any isomorphism \( \gamma : L \to L' \) over \( F \) had \( L' = L \). The restriction of \( \tau \) to \( L \) is such an isomorphism so \( L' = L \).

   (b) Show that \( (\tau \sigma \tau^{-1})(l) = l \).
   \textbf{Solution:} Set \( \tau(l) = l' \) so \( l' \in L \). So \( \sigma(l') = l' \). So \( \tau^{-1}(l') = l \).
   That is \( (\tau \sigma \tau^{-1})(l) = l \).

   (c) From the above show that \( \Gamma[K:L] \) is a normal subgroup (get out those Algebra I notes!) of \( \Gamma[K:F] \). (Assume its already been shown that it is a subgroup. You only need show the normal part.)
   \textbf{Solution:} We’ve shown that if \( \tau \in \Gamma[K:F] \) and \( \sigma \in \Gamma[K:L] \) then \( \tau \sigma \tau^{-1} \) fixes all elements of \( L \) so \( \tau \sigma \tau^{-1} \in \Gamma[K:L] \). That is what we need to show that you have a normal subgroup.

   (d) In the case \( F = \mathbb{Q} \), \( L = \mathbb{Q}(\omega) \), \( K = F(\alpha, \omega) \) (with \( \alpha = 2^{1/3}, \omega = e^{2\pi i/3} \) as in our standard example) give the groups \( \Gamma[K:L] \) and \( \Gamma[K:F] \) explicitly in terms of permutations of \( \alpha, \beta = \alpha \omega, \gamma = \alpha \omega^2 \).

   \textbf{Solution:} \( \Gamma[K:F] \), as done in class, is all six permutations of \( \alpha, \beta, \gamma \). Now fix \( L \), so \( \tau(\omega) = \omega \). There are three cases. Either \( \tau(\alpha) = \alpha \) so \( \tau = e \). Or \( \tau(\alpha) = \beta = \alpha \omega \). Then \( \tau(\beta) = \tau(\alpha \omega) = \alpha \omega = \gamma \) and similarly \( \tau(\gamma) = \alpha \). So \( \tau \) sends \( \alpha, \beta, \gamma \) into \( \beta, \gamma, \alpha \) respectively. Similary if \( \tau(\alpha) = \gamma, \tau \) sends \( \alpha, \beta, \gamma \) into \( \gamma, \alpha, \beta \) respectively.

2. Let \( F \subset K \) be fields in \( \mathbb{C} \) with \([K:F] = 2\). Prove that \( K : F \) is a normal extension.
   \textbf{Solution:} Let \( K : F \) be an extension of degree 2. Then \( K = F(\alpha) \) for some \( \alpha \), where the minimal polynomial of \( \alpha \) in \( K[x] \) has degree 2. Let \( m(x) \in K[x] \) be the minimal polynomial of \( \alpha \), so

   \[ m(x) = x^2 + bx + c \]

   for some \( b, c \in K \). (We may assume \( m(x) \) is monic to make the calculations easier.) We will show \( K : F \) is normal by showing that \( K \)
is a splitting field for the polynomial \( m(x) \) over \( F \). By the quadratic formula, the roots of \( m(x) \) are given by

\[
x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}
\]

As \( m(x) \) is irreducible the roots are distinct. Suppose

\[
\alpha = \frac{-b + \sqrt{b^2 - 4c}}{2},
\]

(the case where the square root is subtracted is similar) then

\[
\beta = \frac{-b - \sqrt{b^2 - 4c}}{2} = \frac{b - \sqrt{b^2 - 4c} - 2b}{2} = -\alpha - b.
\]

Therefore, \( \alpha, \beta \in F(\alpha) = K \). (Another approach: Letting \( \alpha, \beta \) be the roots, \( \alpha + \beta = -b \) immediately from the connection between the roots of a polynomial and its coefficients.)

3. Let \( K_1, K_2 \) be normal extensions of \( Q \). Let \( M \) denote the minimal field containing \( K_1 \cup K_2 \). Prove that \( M \) is a normal extension of \( Q \). [One approach: Write \( K_1 = Q(\alpha_1, \ldots, \alpha_r) \) where the \( \alpha_i \) are all the roots of some \( p(x) \in Q[x] \) and do similarly for \( K_2 \).]

**Solution:** Write \( K_1 \) as above and similarly \( K_2 = Q(\beta_1, \ldots, \beta_s) \) where the \( \beta_j \) are all the roots of some \( q(x) \in Q[x] \). Then \( M = Q(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \) which are all the roots of \( p(x)q(x) \) which is in \( Q[x] \). (Reminder: To be a splitting field you don’t need to have an irreducible polynomial!)

4. Let \( \alpha \) be a root of \( f(x) = x^3 + x^2 - 2x - 1 \in Q[x] \).

(a) Show \( f(x) \) is irreducible over \( Q \).

**Solution:** It is a cubic and by Gauss’s Theorem if it factors it would factor into \( \mathbb{Z}[x] \) so it would a factor \( x - a \) so it would have an integer root \( a \). Then \( a \) would have to divide \(-1\) so it would be either \(+1\) or \(-1\), and neither of them work.

(b) Find \( [Q(\alpha) : Q] \).

**Solution:** \( f(x) \) is irreducible over \( Q \) and \( \alpha \) is a root of \( f(x) \), so \( f(x) \) is the minimal polynomial for \( \alpha \) over \( Q \). Hence \( [Q(\alpha) : Q] = \deg(f) = 3 \).

(c) Set \( \beta = -1/(\alpha + 1) \). Find \( \beta \) in the form \( a + b\alpha + c\alpha^2 \).

**Solution:** First lets find \( 1/(1 + \alpha) \): \( (\alpha + 1)(a + b\alpha + c\alpha^2) = \)
\[ a + b \alpha + c \alpha^2 + a \alpha + b \alpha^2 + c(-\alpha^2 + 2 \alpha + 1) = 1 \]
gives the equations:
\[ a + c = 1, \quad b + a + 2c = 0, \quad c + b - c = 0 \]
so \( b = 0, \ a = 2, \ c = -1 \). Thus \( \beta \) is the negative of that:
\[ \beta = \alpha^2 - 2. \]

(d) Show that \( f(\beta) = 0 \). (Bit of grunt work here!)
Solution: We need show
\[
\frac{-1}{(1 + \alpha)^3} + \frac{1}{(1 + \alpha)^2} - 2 - \frac{1}{1 + \alpha} - 1 = 0
\]
Multiplying out by \((1 + \alpha)^3\) we get that a polynomial which is zero. Or, we need show
\[
(\alpha^2 - 2)^3 + (\alpha^2 - 2)^2 - 2(\alpha^2 - 2) - 1 = 0
\]
One makes a list of \( \alpha^3, \alpha^4, \alpha^5, \alpha^6 \) and everything works out.

(e) Find \( \gamma \in Q(\alpha) \), \( \gamma, \beta, \alpha \) distinct, with \( f(\gamma) = 0 \). (Idea: If \( f(x) = (x - \alpha)(x - \beta)(x - \gamma) \) then \( \alpha + \beta + \gamma \) is determined.)
Solution: \( \alpha + \beta + \gamma = -1 \) so \( \gamma = -1 - \alpha - \beta \).

(f) Deduce that \( Q(\alpha) : Q \) is normal.
Solution: It is the splitting field of \( f(x) \).

(g) List all of the \( Q \)-automorphisms of \( Q(\alpha) \). What familiar group is \( \Gamma(Q(\alpha) : Q) \) isomorphic to?
Solution: When \( [Q(\gamma) : Q] = r \) and \( Q(\gamma) \) is normal over \( Q \) then the Galois group has precisely \( r \) elements. But there is precisely one group on three elements. So \( \Gamma(Q(\alpha) : Q) \) must be isomorphic to \( Z_3 \). One is the identity. Then there is a \( \sigma \) with \( \sigma(\alpha) = \beta \). We can’t have \( \sigma(\gamma) = \gamma \) as that would make \( \sigma \) the identity (as \( Q(\alpha) = Q(\gamma) \)) so \( \sigma(\gamma\alpha\beta) = \alpha \) and \( \sigma(\beta\gamma) = \gamma \). The final automorphims sends \( \alpha, \beta, \gamma \) to \( \alpha, \gamma, \beta \) respectively.

(h) Let \( K \) be the fixed field of \( \Gamma(Q(\alpha) : Q) \). Prove \( K = Q \).
Solution: This is the Galois Correspondence Theorem – the entire Galois Group corresponds to the ground field.

5. As in last week’s assignment set \( \alpha = 2^{1/4}, \beta = i\alpha, \gamma = -\alpha, \delta = -i\alpha \).
Set \( p(x) = x^4 - 2 \). Set \( K = Q(\alpha, \beta, \gamma, \delta) \). Set \( L = Q(i) \). Also, set \( M = Q(\alpha) \) and \( N = Q(\sqrt{2}) \).

(a) Give the factorization of \( p(x) \) into irreducible factors in \( Q[x] \).
Solution: It is irreducible by the Eisenstein Criterion.

(b) Give the factorization of \( p(x) \) into irreducible factors in \( K[x] \).
Solution: It completely factors \( p(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \).
(c) Give the factorization of $p(x)$ into irreducible factors in $M[x]$.
Solution: Well, $(x - \alpha)$ is a factor. But as $\gamma = -\alpha \in M$ so is $(x - \gamma) = (x + \alpha)$. Taking those out $p(x) = (x - \alpha)(x + \alpha)(x^2 + \sqrt{2})$. Here $\sqrt{2} = \alpha^2 \in M$. Also, $x^2 + \sqrt{2}$ is irreducible in $M[x]$ as its two roots, $\gamma, \delta$, are not in $M$ as they are complex numbers.

(d) Give the factorization of $p(x)$ into irreducible factors in $N[x]$.
Solution: $p(x) = (x^2 - \sqrt{2})(x^2 + \sqrt{2})$. As $[Q(\alpha) : Q] = 4$, $\alpha \notin N$. Thus $(x - \alpha)$ is not a factor in $N[x]$. Similarly $x - \beta, x - \gamma, x - \delta$ are not factors. Thus the quadratics above are irreducible in $N[x]$. 