Here we show how to solve the general quartic equation

\[ f(x) = x^4 + ax^3 + bx^2 + cx + d = 0 \]  

though we won’t fully do it. Galois Theory will play a key role.

Call the roots \( \alpha, \beta, \gamma, \delta \) and set \( K = Q(\alpha, \beta, \gamma, \delta) \), the splitting field. We’ll assume the Galois Group \( \Gamma(K : Q) \) is \( S_4 \), the full permutation group on \( \alpha, \beta, \gamma, \delta \). (When it isn’t some things collapse and the argument is easier.)

Indeed, combining Galois Theory and the Sylow Theorems there will be a way of solving it. The Sylow Theorem gives a subgroup \( H = H_1 \) of \( S_4 \) with eight elements and then inside it there is a subgroup \( H_2 \) with four elements, then inside that a subgroup \( H_3 \) with two elements. The Galois Correspondence Theorem then takes the descending tower of groups \( S_4 \supset H_1 \supset H_2 \supset H_3 \supset \{e\} \) and corresponds an ascending tower of fields \( Q \subset K_1 \subset K_2 \subset K_3 \subset K \). Then \( [K_1 : Q] = 3 \) so writing \( K_1 = Q(\rho) \), \( \rho \) satisfies a cubic which can be solved by Cardano’s formula. The remaining extensions are all of dimension two and so they can be solved in terms of square roots. So the theory says that there will be a solution to \( f(x) = 0 \) in terms of various square and cube roots.

Let’s do this more explicitly. There are actually three (conjugate) subgroups of \( S_4 \) of size eight. One, call it \( H_1 \), consists of \( e \), flipping \( \alpha, \beta \) and/or flipping \( \gamma, \delta \) (thats three) and flipping \( \{\alpha, \beta\} \) and \( \{\gamma, \delta\} \) (four, e.g., \( \alpha \to \gamma, \beta \to \delta, \gamma \to \beta, \delta \to \alpha \)). Its a sweet non-Abelian group with eight elements. (A nice exercise is to show that it is isomorphic to the Dihedral Group on eight elements.) Inside that we can take \( H_2 \) as the group of doubleflips: \( e, (\alpha\beta)(\gamma\delta), (\alpha\gamma)(\beta\delta), (\alpha\delta)(\beta\gamma) \). This is the Viergruppe\(^1\)

\(^1\)traditional German term, meaning Four Group

Everything squared is the identity so it is isomorphic to \( (Z_2 \times Z_2, +) \). Inside of that we can take simply \( H_3 = \{e, (\alpha\beta)(\gamma\delta)\} \).

It will help to set

\[ \kappa = \alpha\beta + \gamma\delta \]  
\[ \lambda = \alpha\gamma + \beta\delta \]  
\[ \mu = \alpha\delta + \beta\gamma \]

Then \( \kappa \) is fixed by the group \( H_1 \) of eight permutations given above. But the other permutations \( \sigma \in S_4 \) send \( \kappa \) to \( \lambda \) and \( \mu \). Indeed, the \( \sigma \in S_4 \) with \( \sigma(\kappa) = \lambda \) is a right coset of \( H \). Take any \( \tau \in S_4 \) with \( \tau(\kappa) = \lambda \). Then \( \sigma(\kappa) = \lambda \) if and only if \( \sigma \in H\tau \). We therefore have \( Q(\kappa)^\tau = H \) so Galois
Correspondence gives \( Q(\kappa) : Q = 24/8 = 3 \). We can be more explicit and write

\[
g(x) = (x - \kappa)(x - \lambda)(x - \mu) \tag{5}
\]

The coefficients of \( g(x) \) are the elementary symmetric polynomials of \( \kappa, \lambda, \mu \). Any \( \sigma \) on \( \alpha, \beta, \gamma, \delta \) permutes the \( \kappa, \lambda, \mu \) and hence fixes any symmetric polynomial of them. Hence \( g(x) \in Q[x] \). Indeed, with a lot of work, one can get the coefficients of \( g(x) \) explicitly in terms of \( a, b, c, d \).

We solve \( g(x) = 0 \) by Cardano’s formula, yielding the \( \kappa, \lambda, \mu \). In terms of fields, set \( K_1 = Q(\kappa) \), \( K_2 = K_1(\lambda) = Q(\kappa, \lambda, \mu) \). Then \( Q \subset K_1 \subset K_2 \), and \( [K_2 : Q] = 6 \), so that \([K : K_2] = 4 \). Further \( K_2^* \) is precisely the \( \sigma \in S_4 \) which fix both \( \kappa \) and \( \lambda \) which is \( H_2 \), the doubleflips.

Now set

\[
\epsilon = \alpha \beta - \gamma \delta \tag{6}
\]

Then \( \kappa^2 - \epsilon^2 = 4\alpha \beta \gamma \delta = 4d \) so that \( \epsilon = (\kappa^2 - 4d)^{1/2} \). Indeed, setting \( K_3 = K_2(\epsilon) \), \( K_3^* \) consists of solely \( \epsilon \) and the doubleflip \((\alpha \beta)(\gamma \delta)\). That is, \( K_3^* = H_3 \).

As \([K : K_3] = 2 \), \( K = K_3(\sqrt{\zeta}) \) for some \( \zeta \in K_3 \) and so any \( \xi \in K \), including \( \xi = \alpha \), and be expressed as \( \theta + \theta \sqrt{\zeta} \) for some \( \theta_1, \theta_2 \in K_3 \).

Getting an explicit \( \zeta \) is a challenge. (And just imagine you were a fifteenth century mathematician before Galois Theory had been invented!) Add to \( (6) \) the analogous two:

\[
\theta = \alpha \gamma - \beta \delta \tag{7}
\]

\[
\eta = \alpha \delta - \beta \gamma \tag{8}
\]

so that

\[
\kappa^2 - \epsilon^2 = \lambda^2 - \theta^2 = \mu^2 - \eta^2 = 4d \tag{9}
\]

or

\[
\epsilon^2 = \kappa^2 - 4d, \theta^2 = \lambda^2 - 4d, \eta^2 = \mu^2 - 4d \tag{10}
\]

Consider \( K_4 = K_3(\theta) = K_2(\epsilon, \theta) \). As \( \lambda \in K_2 \) this is an extension by a square root. So the theory says that \( K_4 \) is the whole splitting field. So where is \( \alpha \)? To find it explicitly we set

\[
w = \epsilon \theta \eta \tag{11}
\]

The eight products split into

\[
[\alpha^3 \beta \gamma \delta + \alpha \beta^3 \eta \delta + \alpha \beta \gamma^3 \delta + \alpha \beta \gamma \delta^3] - [\alpha^2 \beta^2 \gamma \delta + \alpha^2 \beta^2 \gamma^2 + \alpha^2 \gamma^2 \delta^2 + \beta^2 \gamma^2 \delta^2] \tag{12}
\]

So \( w \) is a symmetric polynomial of \( \alpha, \beta, \gamma, \delta \) and therefore a function of the elementary symmetric polynomials of \( \alpha, \beta, \gamma, \delta \) which are \(-a, b, -c, d\) and hence \( w \in Q \). We shall take \( \zeta = \lambda^2 - 4d \) so that \( \sqrt{\zeta} = \theta \). Then \( K = K_3(\theta) = K_2(\epsilon, \theta) \). But as \( \epsilon \theta \eta \in Q \), \( K = K_2(\epsilon, \theta, \eta) \). So \( K \) contains \( \kappa, \epsilon, \lambda, \theta, \mu, \eta \). Hence it contains \( \frac{1}{2}(\kappa \pm \epsilon), \frac{1}{2}(\lambda \pm \theta), \frac{1}{2}(\mu \pm \eta) \). These are the six products \( \alpha \beta, \alpha \gamma, \alpha \delta, \beta \gamma, \beta \delta, \gamma \delta \). Taking ratios appropriately \( K \) contains all ratios of any two of the roots. In particular \( \beta = \frac{\beta \gamma}{\alpha}, \gamma = \frac{\gamma \delta}{\alpha \delta}, \) and \( \delta = \frac{\beta \delta}{\beta \alpha} \) are all in \( K \). Then

\[
\alpha = \frac{\alpha + \beta + \gamma + \delta}{1 + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} + \frac{\delta}{\alpha}} \tag{13}
\]
The numerator is simply $-a$. The denominator is in $K$ as each addend is in $K$. Thus $\alpha$ is given explicitly (if you are a fifteenth century monk with sufficient time!) in $K$. 