Pell’s Equation

Throughout, \(d\) is a positive integer which is not a square. \(d = 2, 3\) provide good examples. Pell’s Equation is

\[
x^2 - dy^2 = \pm 1
\]  

This is a Diophantine equation, by which we mean that we are looking for solutions in the integers \(\mathbb{Z}\). But our approach is through irrational numbers!

Set

\[
Z[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}
\]  

(2)

\(Z[\sqrt{d}]\) (exercise!) is a ring. What are its units \(\alpha = x + y\sqrt{d}\)? Well,

\[
\alpha^{-1} = \frac{1}{x + y\sqrt{d}} = \frac{1}{x + y\sqrt{d}} \frac{x - y\sqrt{d}}{x^2 - dy^2} = \frac{x - y\sqrt{d}}{x^2 - dy^2}
\]  

(3)

If \(x^2 - dy^2 = \pm 1\) then \(\alpha^{-1} \in Z[\sqrt{d}]\) so \(\alpha\) is a unit. Conversely, if \(\alpha\) is a unit then we must have \((x^2 - dy^2)|x\) and \((x^2 - dy^2)|y\) so \((x^2 - dy^2)^2|x^2\) and \((x^2 - dy^2)^2|y^2\) so \((x^2 - dy^2)|1\) and \(x^2 - dy^2 = \pm 1\).

That is, the solutions \(x, y\) to Pell’s Equation (1) correspond precisely to the units \(\alpha = x + y\sqrt{d}\) of \(Z[\sqrt{d}]\).

Its easy to show that there are no units when \(x = 0\) and \(x = \pm 1\) are the only units when \(y = 0\). Suppose \(\alpha = x + y\sqrt{d} > 1\) is a unit. Lets examine the four possible signs of the coefficients. Say \(a, b\) are positive.

**Case I:** \(\alpha = -a - b\sqrt{d}\). But thats negative. Fuhgettiboutit.

**Case II:** \(\alpha = a - b\sqrt{d}\). Then \(a + b\sqrt{d} > 1\). So

\[
\pm 1 = a^2 - db^2 = (a + b\sqrt{d})(a - b\sqrt{d}) > 1
\]  

(4)

Fuhgettiboutit.

**Case III:** \(\alpha = -a + b\sqrt{d}\). Then \(a + b\sqrt{d} > 1\). So

\[
\pm 1 = -a^2 + db^2 = (-a + b\sqrt{d})(a + b\sqrt{d}) > 1
\]  

(5)

Fuhgettiboutit.

**Case IV:** \(\alpha = a + b\sqrt{d}\). Gotta be this, only case left!

**Fact:** There exist units \(\gamma > 1\).

This isn’t easy to prove. We’ll assume it and get some nice consequences. For particular \(d\) we merely find some \(\alpha\). For example, \(1 + \sqrt{2}\) with \(d = 2\) and \(2 + \sqrt{5}\) with \(d = 5\). (Sometimes its not so easy! Check out Wiki. For \(d = 13\), \(\alpha = 649 + 180\sqrt{13}\).)
**Theorem:** There exists a least unit $\beta > 1$.

Why? Well, by **Fact** there is some unit $\gamma$. Since $\beta = a + b\sqrt{d}$ must be in Case IV we only need to check the *finite* number of $1 \leq a \leq \beta$. So the infimum will indeed be a minimum.

OK. Let $\beta$ be this least unit.

**Theorem:** Every unit $\kappa > 1$ has $\kappa = \beta^n$ for some positive integer $n$.

**Proof:** The intervals $[\beta^n, \beta^{n+1})$ have union (over $n \geq 1$) $[\beta, \infty)$ so that $\beta^n \leq \kappa < \beta^{n+1}$ for some $n$. But $\beta^{-n}$ is a unit so $\kappa \beta^{-n}$ is a unit and $1 \leq \kappa \beta^{-n} < \beta$ so by the minimality, $\kappa \beta^{-n} = 1$ so $\kappa = \beta^n$.

**Corollary:** Every unit $\kappa$ has $\kappa = \pm \beta^n$ for some integer $n$.

**Proof:** If $0 < \kappa < 1$ apply the Theorem to $\kappa^{-1}$. Then if $\kappa$ is negative apply this to $-\kappa$.

Once we have found $\beta$ we now have all the solutions to Pell’s Equation. For example, with $d = 2$, $1 + \sqrt{2}$ is a unit and we check that $a + b\sqrt{2}$ with $0 < a, b \leq 1$ has no other units (well, here there is only one case but sometimes there are many) so $\beta = 1 + \sqrt{2}$ is the minimal unit. Thus all solutions to (1) with $x, y \geq 1$ (the others are simply gotten by replacing $x$ and/or $y$ by its negative) are given by $x + y\sqrt{2} = \beta^n$. The powers of $\beta$ go $1 + \sqrt{2}, 3 + 2\sqrt{2}, 7 + 5\sqrt{2}, 17 + 12\sqrt{2}, 41 + 29\sqrt{2}, 99 + 70\sqrt{2}$, etc., and note that, indeed, $99^2 = 9801$ is one more than twice $70^2$. 