OLD ALGEBRA FINAL

Please Note: When this final was given there was only one Algebra I, there was no Honors Algebra. You should expect that your final will contain some more difficult problems.

No books or notes. Problems marked (-) are relatively easy while those marked (*) are particularly challenging. Maximal Grade 205.

1. (15) Set \( H = \{ \sigma \in S_8 : \sigma(1) = 1 \} \). Is \( H \) a normal subgroup of \( S_8 \).
   (You can assume without proof that it is a subgroup.) You must give a clear argument for your answer.

2. (10) In \( \mathbb{Z}[i] \) is \( 9 + 4i \in (2 + i) \)? Give your reason!

3. (20) Let \( R \) be a Euclidean Ring. Prove that every ideal of \( R \) is a principal ideal. (Partial credit for correctly defining Euclidean Ring.)

4. (25) With \( D_{10} \) as given in the attached sheet:
   (a) (-)(5) What is the inverse of \( R2 \)?
   (b) (10) Find the conjugacy class of \( R3 \).
   (c) (5) Find \( (R3)^{201} \).
   (d) (-)(5) Is \( D_{10} \) Abelian? Give a short reason.

5. (15) In \( \mathbb{Z}[i] \) find \( q, r \) with \( (8 + 9i) = (3 + 2i)q + r \) and either \( r = 0 \) or \( d(r) < d(3 + 2i) \). (Here \( d(a + bi) = a^2 + b^2 \).) Show all work.

6. (10) Let \( G, g \in G \) and define \( \phi : G \to G \) by \( \phi(x) = gxg^{-1} \). Show that \( \phi \) is a homomorphism.

7. (25) Let \( a, b \) be relatively prime integers, both nonzero. In \( \mathbb{Z}[i] \), set \( M = (a + bi, a - bi) \).
   (a) (10) Prove \( 2a \in M \). Prove \( 2b \in M \). Prove \( 2 \in M \).
   (b) (5) Prove \( M \neq (2) \).
   (c) (10) Now further assume \( a, b \) are both odd. Find, with proof, \( \kappa \) so that \( M = (\kappa) \).

8. (30) (This problem has many parts. If you miss part, don’t stop! You can assume previous parts in doing later parts.) Set \( \mathbb{Z}[\sqrt{-2}] = \{ a + b\sqrt{-2} : a, b \in \mathbb{Z} \} \)
   Further, for \( \alpha = a + b\sqrt{-2} \in \mathbb{Z}[\sqrt{-2}], \alpha \neq 0, \) define \( d(\alpha) = a\overline{\alpha} \) where \( \overline{\alpha} \) is the complex conjugate.
(a) (5) Draw a nice picture of the points of $\mathbb{Z}[\sqrt{-2}]$ on the complex plane.

(b) (5) Set $U = (1 + \sqrt{-2})$. On your picture above, encircle the points belonging to $U$ so that a clear pattern emerges.

(c) (5) Give a set of coset representatives for $\mathbb{Z}[\sqrt{-2}]/U$. Short reason, please.

(d) (5) Show that $d(\alpha \beta) = d(\alpha)d(\beta)$ for all nonzero $\alpha, \beta \in \mathbb{Z}[\sqrt{-2}]$.

(e) (*) (5) Show that 7 is a prime in $\mathbb{Z}[\sqrt{-2}]$. (Use 8d)

(f) (10) Let $\alpha, \beta \in \mathbb{Z}[\sqrt{-2}]$, $\beta \neq 0$. Prove that there exist $q, r \in \mathbb{Z}[\sqrt{-2}]$ with $\alpha = q\beta + r$ and either $r = 0$ or $d(r) < d(\beta)$.

9. (15) The center $Z$ of a group $G$ is the set of all $z \in G$ with the property that $zg = gz$ for all $g \in G$. Prove that $Z$ is a subgroup of $G$.

10. (15) Give definitions of the terms:

   (a) (5) $U$ is an ideal of $R$

   (b) (5) $H$ is a normal subgroup of $G$.

   (c) (5) $\phi : R \to S$ is a homomorphism. ($R, S$ rings.)

11. (15) Set $F = \mathbb{Z}_2[x]/(x^5 + x^2 + 1)$. Assume as a fact that $x^5 + x^2 + 1 \in \mathbb{Z}_2[x]$ is irreducible.

   (a) (-) (5) How many elements are in $F$?

   (b) (*) (10) Find $x^{31000002}$ in $F$. (Hint: There is a shortcut!)

12. (10) List all the Abelian groups with $p^3$ elements. ($p$ prime) (No proof required!)
The Dihedral Group $D_{10}$ is the group of symmetries of the regular 5-gon. We imagine the vertices of the regular 5-gon labelled 0, 1, 2, 3, 4 in counterclockwise direction.

The symmetries come in three forms:

1. $R_i, i = 1, 2, 3, 4$. This is a rotation by $i$ notches in the counterclockwise direction.

2. $F_i, i = 0, 1, 2, 3, 4$. This is a flip around point $i$.

3. The identity $e$.

A Table for the Dihedral Group $D_{10}$.

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