HONORS ALGEBRA MIDTERM

Do all problems. Maximal Score: 100. Problems marked with (*) are more difficult.

Notations: $\mathbb{R}$ is the reals, $\mathbb{Z}$ the integers, $\mathbb{Q}$ the rational numbers. $\mathbb{R}^*$ is the nonzero reals under multiplication. $\mathbb{R}^+$ is the positive reals under multiplication. $S_n$ is the set of permutations of $1, \ldots, n$. $GL_n(\mathbb{R})$ is the set of nonsingular $n \times n$ matrices with real coefficients, under matrix multiplication. $a \sim b$ means there exists $g, g^{-1}ag = b$. $C(a) = \{b : a \sim b\}$, $c(a) = |C(a)|$, $N(a) = \{x : ax = xa\}$.

1. (10) Let $\phi : G \to H$ be a homomorphism. Let $g \in G$, $h = \phi(g) \in H$. Let $o(g) = a, o(h) = b$, both finite. Prove $b \mid a$.

Solution: $h^a = \sigma(g)^a = \sigma(g^a) = \sigma(e) = e$. As $h$ has order $b$ the powers of $h$ that are the identity are precisely the multiples of $b$. In particular, $a$ must be a multiple of $b$.

2. (10) Let $A \in GL_n(\mathbb{R})$. Suppose $\lambda$ is a real eigenvalue of $A$. Suppose $A \sim B$. Prove $\lambda$ is a real eigenvalue of $B$.

Solution: For some $\vec{v} \neq \vec{0}$, $A\vec{v} = \lambda\vec{v}$. As $A \sim B$ there is a nonsingular $P$ with $B = P^{-1}AP$. Set $\vec{w} = P^{-1}\vec{v}$. Then $P\vec{w} = \vec{v}$, $A\vec{w} = \lambda\vec{w}$, $P^{-1}[\lambda\vec{w}] = \lambda P^{-1}[\vec{v}] = \lambda\vec{w}$ so $B\vec{w} = \lambda\vec{w}$.

3. (20) Let $n \geq 4$ and set $\sigma = (123) \in S_n$.

(a) (5) Express $\sigma$ as the product of transpositions. Is $\sigma$ even or odd?

Solution: $\sigma = (12)(13)$ is even.

(b) (5) Find $c(\sigma)$.

Solution: $C(\sigma)$ is all $(xyz)$. There are $n(n-1)(n-2)$ choices for $x, y, z$ but three of them $(xyz), (yzx), (zxy)$ are the same so $c(\sigma) = n(n-1)(n-2)/3$.

(c) (5) Find $|N(\sigma)|$.

Solution: $|N(\sigma)| = n!/[n(n-1)(n-2)/3] = 3(n-3)!$

(d) (*) (5) Describe all $\tau \in N(\sigma)$.

Solution: $\tau$ sends 1, 2, 3 to either 1, 2, 3 or 2, 3, 1 or 3, 1, 2 and permutes 4, $\ldots$, $n$ in any of the $(n-3)!$ ways.

4. (20) Some definitions, please. (Assume group and subgroup are already defined.)

(a) (5) Let $\phi : G \to H$ be a homomorphism. Define the kernel $K_\phi$.

Solution: $K_\phi = \{g \in G : \phi(g) = e\}$
(b) (5) Define: $H$ is a normal subgroup of $G$.
Solution: $H$ is a subgroup and for all $h \in H$, $g \in G$ we have $g^{-1}hg \in H$.

(c) (5) Let $g \in G$. Define the order $o(g)$.
Solution: $o(g)$ is the least positive integer $n$ such that $g^n = e$.

(d) (5) Define: $G$ is a simple group.
Solution: $G$ is a group with no nontrivial normal subgroups. That is, $\{e\}$ and $G$ itself are the only normal subgroups of $G$.

5. (10)(*) Let $G = Re$ and $H = Q$, both under addition. Prove that, other than the identity, no element of $G/H$ has finite order. (For partial credit show that no $r \in G/H$ has order two.)

Solution: Any element of $G/H$ can be written $\overline{r}$ for some $r \in Re$. If $\overline{r}$ has order $n$ then $0 = nr = n\overline{r}$. Thus $nr \in Q$. But then $r \in Q$ so $\overline{r} = \overline{0}$.

6. (*) (15) Let $\phi : G \rightarrow H$ be a surjective homomorphism. Set $K = K_{\phi}$, the kernel of $\phi$. Prove that $G/K \cong H$. (You can use the proof given in class or come up with your own proof. For partial credit define the isomorphism $\Psi : G/K \rightarrow H$.)

Solution: As done in class, or in the text.

7. (15) In $S_4$ set $\sigma = (12)(34)$, $\tau = (13)(24)$, $\gamma = (14)(23)$, $e$ the identity.

Let $H = \{e, \sigma, \tau, \gamma\}$. Let $H_1 = \{e, \sigma\}$. You may assume $H$ and $H_1$ are groups.

(a) (5) Prove $H_1$ is a normal subgroup of $H$

Solution: $H_1$ has 2 elements and $H_2$ has 4 elements. Any subgroup with precisely half the elements is normal.

(b) (5) Prove $H$ is a normal subgroup of $S_4$.

Solution: The conjugates of $\sigma, \tau, \gamma$ are those elements with the same cycle lengths, i.e., precisely two two cycles. These are themselves.

(c) (5) Prove $H_1$ is NOT a normal subgroup of $S_4$

Solution: $\tau \not\in H$ is a conjugate of $\sigma$.

(FYI: This is a good example of how a normal subgroup of a normal subgroup need not be a normal subgroup.)