ALGEBRA MIDTERM

God created the integers; the rest is the work of man.
– Kronecker

No books or notes. Do all problems. Maximum score: 105.

1. (20) Prove that a Euclidean Domain is a Principle Ideal Domain.

2. (10) In \( \mathbb{Z}[i] \) set \( \alpha = 29 + 4i \) and \( \beta = 2 + 3i \). Find \( q, r \in \mathbb{Z}[i] \) with \( \alpha = q\beta + r \) and either \( r = 0 \) or \( d(r) < d(\beta) \). (Here \( d(a+bi) = a^2 + b^2 \).)

3. (10) State clearly Eisenstein’s Criterion for \( a_0 + a_1x + \ldots + a_nx^n \in \mathbb{Z}[x] \) to be irreducible in \( \mathbb{Z}[x] \). Apply it to show \( x^7 - 300 \) is irreducible in \( \mathbb{Z}[x] \).

4. (10) Let \( \epsilon = e^{2\pi i/7} \). Find (with short reason) \( [\mathbb{Q}(\epsilon) : \mathbb{Q}] \) and give a basis for \( \mathbb{Q}(\epsilon) \) as a vector space over \( \mathbb{Q} \).

5. (10) Let \( [K : F] = n \) and suppose \( \alpha_1, \ldots, \alpha_n \in K \) form a basis for \( K \) as a vector space over \( \mathbb{Q} \). Let \( \beta \in K, \beta \neq 0 \). Prove \( \beta\alpha_1, \ldots, \beta\alpha_n \) form a basis for \( K \) as a vector space over \( \mathbb{Q} \).

6. (10) Assume as a fact that \( x^5 + x^2 + 1 \) is irreducible in \( \mathbb{Z}_2[x] \). (It is!) Set \( F = \mathbb{Z}_2[x]/(x^5 + x^2 + 1) \).

   (a) (5) How many elements are in \( F \). Describe the elements.

   (b) (5) Find \( x^{3100005} \) in \( F \). Your answer should be one of the elements you described in (6a)
7. (20) Let $p$ be an integer prime of the form $p = 4k + 1$. (Note: You can assume earlier parts when working on later parts of this problem.)

(a) (5) Show that there exists $a \in \mathbb{Z}_p^*$ with $a^2 = -1$ in $\mathbb{Z}_p$. (Hint: Let $g$ be a generator.)

(b) (5) Let $a$ be a positive integer. Show that $a^2 + 1$ is not a prime in the Gaussian Integers.

(c) (5) Using (7a,7b) show that $p$ is not a prime in the Gaussian Integers.

(d) (5) Using (7c) show that there exist integers $x, y$ with $p = x^2 + y^2$.

Note: This result is called the Fermat Two Squares Theorem.

8. (5) Let $F \subset K \subset L$. State (do not prove!) the relationship between $[K : F]$, $[L : K]$ and $[L : F]$. (You can assume they are all finite.)

9. (10) Let $F = \mathbb{Q}[x]/(x^3 - x - 1)$. Write $x^i$ in the form $ax^2 + bx + c$ for $i = 3, 4$ and for $i = 9$. (For the last there is a shortcut!)

Every block of stone has a statue inside it, and it is the task of the sculptor to discover it. – Michelangelo