The voyage of discovery lies not in seeking new horizons, but in seeking with new eyes.
– Proust

1. Let $R$ be a ring. Call $a \in R$ a unit if $ab = 1$ for some $b \in R$. Let $X$ be the set of units. Prove that $X$ forms a group under multiplication. What is our standard notation for $X$ in the case where $R = \mathbb{Z}_n$?

2. Recall $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Find a unit $\alpha = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ which has $a \geq 10$. (One approach: Find some unit $\beta$ other than $\pm 1$. Then, as the units form a group, any power $\beta^w$ is also a unit.)

3. Let $R$ be a ring of characteristic 3.

   (a) Prove that the map $\phi : R \to R$ given by $\phi(x) = x^3$ is a ring homomorphism.

   (b) Assume further that $R$ is an Integral Domain. Now prove that $\phi$ is injective.

   (c) Assume yet further that $R$ is finite. Now prove $\phi$ is an isomorphism.

4. Give a natural set of representatives for $\mathbb{Z}[i]/(3)$. Give the addition and multiplication tables for $\mathbb{Z}[i]/(3)$. Give the multiplicative inverse (you can read it off your table!) for each nonzero element of $\mathbb{Z}[i]/(3)$.

Fearing error and fearing truth are one and the same. Those who fear making mistakes are incapable of discovery. When we worry about making mistakes, the error within us becomes an unmovable rock. In our fear, we cling to what we have declared to be “true” one day, or what has always been presented as such. When we are driven by a thirst for knowledge, and not by the fear of seeing a false security fade away, then error, like suffering and sadness, passes through us without ever gaining substance, and the trace it leaves is that of renewed knowledge.

Alexander Grothendieck, *Recôles et Semailes*