Honors Algebra I  
Assignment 8  
Due, October 30

She guessed at it all, what might live, moving purposefully or drifting aimlessly, under the deep water around her, but she didn’t think too much about any of it. It was enough to be aware of the million permutations possible around her, and take comfort in knowing she would not, and really could not, know much at all.
Dave Eggers, The Circle

1. Ima Geek is given a group $G$ with $125 = 5^3$ elements and is trying to follow one of the cases of the proof given in class to get $G$ isomorphic to $Z_{25} \times Z_5$. Ms. Geek doesn’t realize that the group actually is $Z_{25} \times Z_5$ and she picks $a = (3, 4)$ as her first element. Show that $a$ has order 25. Set $N = \{0, a, \ldots, 24a\}$. Show that every element of $G/N$ can be written as $(0, k)$ for a unique $0 \leq k < 5$. Now she picks $b = (11, 1)$ as her second element. Find $0 \leq k < 5$ so that $b = (0, k)$. Follow the argument given in class to find an explicit $c \in G$ with $c = b - ia$ for an explicit $i$ so that $o(c) = 5$ in $G$. Find $0 \leq i < 25$ and $0 \leq j < 5$ with $ai + cj = (1, 0)$ in $Z_{25} \times Z_5$.

2. Let $G, H$ be groups under multiplication with identities $e_G, e_H$ respectively. Let $J = G \times H$ be their direct product. Set $\Lambda = \{(g, e_H) \in J : g \in G\}$

(a) Prove $\Lambda$ is a subgroup of $J$.
(b) Prove $\Lambda$ is a normal subgroup of $J$.
(c) Let $n \geq 5$. Prove (using the above and results proven in class) that $S_n$ is not isomorphic to $(S_n/A_n) \times A_n$.

3. Three problems about manipulating products of cyclic groups.

(a) Write $Z_2 \times Z_2 \times Z_2 \times Z_9 \times Z_5 \times Z_{25}$ as the product of cyclic groups $Z_{a_i}$, $1 \leq i \leq s$ (you find the $s$) with $a_i$ dividing $a_{i+1}$ for all $1 \leq i < s$.

(b) Write $Z_4 \times Z_{40} \times Z_{200} \times Z_{1400}$ as the product of cyclic groups of prime power order. (Prime power includes primes themselves.)

(c) Write $Z_5 \times Z_6 \times Z_7$ in both of the above forms.
Nothing is more fruitful - all mathematicians know it - than those obscure analogies, those disturbing reflections of one theory in another; those furtive caresses, those inexplicable discords; nothing also gives more pleasure to the researcher. The day comes when the illusion dissolves; the yoked theories reveal their common source before disappearing. As the Gita teaches, one achieves knowledge and indifference \(^1\) at the same time.

André Weil

\[^1\text{Some translators of the original Sanskrit use “detachment” here.}\]