The mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.
– G.H. Hardy

1. Let $a, b, c, d$ be real numbers. Let $\alpha = a + bi \in \mathbb{C}$. Let $\beta = c + di \in \mathbb{C}$ be such that $\beta^2 = \alpha$. (That is, $\beta$ is one of the two square roots of $\alpha$.) Let $K$ be a field with $\mathbb{Q} \subseteq K$ and $a, b \in K$. Find an explicit tower (you find the $r$!)

$$K = K_0 \subset K_1 \subset \ldots \subset K_r = L$$

with all $[K_{j+1} : K_j] = 2$ and $c, d \in L$. Further, all elements of each $K_i$ must be real. (In particular, you can’t extend by $i$.) (This is a challenging problem. Its helpful to think in terms of polar coordinates and first find $|\beta| = \sqrt{c^2 + d^2}$.)

2. (This problem graded double.) Let $\epsilon = e^{\frac{2\pi i}{5}}$. Set $K = \mathbb{Q}(\epsilon)$.

(a) Find $[K : \mathbb{Q}]$ and a basis for $K$ over $\mathbb{Q}$.

(b) Set $\gamma = \epsilon + \epsilon^4$. Show that $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2$ by finding an explicit quadratic equation, with coefficients in $\mathbb{Q}$, satisfied by $\gamma$. (Note: Since you have the basis this is a linear algebra problem: finding a dependence between 1, $\gamma, \gamma^2$.) Express $\gamma$ in terms of elementary trig functions.

(c) Use the quadratic formula to solve $\gamma$ explicitly. Find an explicit $d \in \mathbb{Z}$ with $\mathbb{Q}(\gamma) = \mathbb{Q}(\sqrt{d})$.

(d) Show $[K : \mathbb{Q}(\gamma)] = 2$. (This is immediate if you see it.)

(e) As $\epsilon \in K$, find an explicit quadratic equation, with coefficients in $\mathbb{Q}(\gamma)$, satisfied by $\epsilon$.

(f) Use the quadratic formula to solve $\epsilon$ explicitly. (Some of the terms will be square roots of non-real numbers, but lets allow that. The object is to write $\epsilon$ in terms of usual field expressions and square roots.)

(g) Write $\epsilon = a + bi$. Find $a, b$ in terms of usual field expressions and square roots, but not involving complex numbers.
3. Let \( f(x) = x^3 + ax^2 + cx + d \in \mathbb{Q}[x] \) be an irreducible cubic with one real root \( \alpha \) and two nonreal roots \( \beta, \gamma \).

(a) Argue that \( \gamma = \overline{\beta} \). (Note: Here, and often, we let \( \overline{\kappa} \) denote the complex conjugate of \( \kappa \).)

(b) Argue that \( \gamma \notin \mathbb{Q}(\alpha) \).

(c) Argue that \( [\mathbb{Q}(\alpha, \gamma) : \mathbb{Q}(\alpha)] = 2 \).

(d) Show that \( \beta \in \mathbb{Q}(\alpha, \gamma) \) and that \( \alpha \in \mathbb{Q}(\beta, \gamma) \).

(e) Argue that \( [\mathbb{Q}(\alpha, \beta, \gamma) : \mathbb{Q}] = 6 \).

(f) Argue that \( \gamma \notin \mathbb{Q}(\beta) \). (Idea: If \( \gamma \in \mathbb{Q}(\beta) \) then show that \( \mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(\beta) \) and get a contradiction.)

Remark: By the last part, \( \mathbb{Q}(\beta) \) is a field which is not closed under complex conjugation.

4. Suppose \( \alpha \in \mathbb{C} \) satisfies the equation

\[
\alpha^3 + \sqrt{2}\alpha^2 - (7^{1/5} - 8)\alpha = \sqrt{8 + 9\sqrt{11}}
\]

Use the Tower Theorem to bound \( [\mathbb{Q}(\alpha) : \mathbb{Q}] \). (Idea: First extend by coefficients of polynomial.)

Dear Sir,

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 25 years of age. I have no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics...I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as “startling”

– Ramanujan