Comstock grins and says, ‘You sound awfully sure of yourself, Waterhouse! I wonder if you can get me to feel that same level of confidence.’

Waterhouse frowns at the coffee mug. ‘Well, it’s all in the math,’ he says, ‘If the math works, why then you should be sure of yourself. That’s the whole point of math.’

from Cryptonomicon by Neal Stephenson

1. Let $G$ denote the set of linear functions $f(x) = mx + b$ on the real line with $m \neq 0$. Denote such a function by $(m, b)$. Define a product $f * g$ as the function $h(x) = g(f(x))$. (Check assignment 1 and the solutions for earlier work on this group.) Recall $C(f), N(f)$ denote the conjugate class and the normalizer of $f$.

   (a) Describe $C(f)$ and $N(f)$ when $m \neq 1$ and $b \neq 0$.
   (b) Describe $C(f)$ and $N(f)$ when $m \neq 1$ and $b = 0$.
   (c) Describe $C(f)$ and $N(f)$ when $m = 1$ and $b \neq 0$.
   (d) Describe $C(f)$ and $N(f)$ when $m = 1$ and $b = 0$.
   (e) Describe $Z[\mathbb{G}]$, the center of the group.

2. Let $\sigma \in S_n$ be (in cycle notation)

   $$\sigma = (12\cdots n)$$

   (a) Describe in words the $\gamma \in C(\sigma)$.
   (b) Find $|C(\sigma)|$.
   (c) Deduce $|N(\sigma)|$.
   (d) (*) Describe $N(\sigma)$ explicitly. (Idea: Since you already know that $N(\sigma)$ has precisely woggle elements, you should look for woggle distinct elements that are in $N(\sigma)$ and then you have them all.)

3. Let $o(G) = p^n$, $p$ prime. Prove that $o(Z[G]) \neq p^{n-1}$. (Idea: Examine the proof that groups of order $p^2$ are Abelian.)

4. Let $G = \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_{125}$. Find the order of $(3, 10, 12)$. Give a good description and a precise count on those $(a, b, c) \in G$ order 25.
5. Let $H$ be a normal subgroup of $G$ (all under multiplication). Let $g \in G$ and let (as usual) $\overline{g}$ denote the corresponding element in $G/H$. Suppose $\overline{g}$ has order $a$ (in $G/H$) and $g^a$ has order $b$ (in $G$). Prove $g$ has order $ab$ in $G$.

6. (Warning: Lots of grunt work here.) Let $H$ be the subgroup of $S_4$ which is the normalizer of $(12)(34)$, as discussed in class. Is this group isomorphic to the group of symmetries of the square, as given in the solutions to Assignment 1. If they are not isomorphic, prove it. If they are isomorphic, give an explicit isomorphism.

The universe is not only queerer than we suppose but queerer than we can suppose.
– J.B.S. Haldane