The universe is not only queerer than we suppose but queerer than we can suppose.
– J.B.S. Haldane

1. Let $F \subset K$, both fields, and consider $K$ as a vector space over $F$. Let $\alpha \in K$, $\alpha \neq 0$. Prove that the map $T_\alpha : K \rightarrow K$ given by $T_\alpha(\beta) = \alpha \beta$ is a homomorphism. That is, show

(a) $T_\alpha(v_1 + v_2) = T_\alpha(v_1) + T_\alpha(v_2)$ for all $v_1, v_2 \in K$

(b) $T_\alpha(\lambda v) = \lambda T_\alpha(v)$ for all $v \in K$, $\lambda \in F$.

(c) Prove further that $T_\alpha$ an isomorphism between $K$ and itself.

2. Presidential Trivia:

(a) Which president had a great stamp collection?

(b) Which was the fattest president?

(c) Which two presidents died on the same day?

(d) Which president was divorced?

3. Let $\alpha \in C$ be a root of $x^3 + x + 3$. (This cubic has no special properties.) Write $\alpha^i$ in the form $a + b\alpha + c\alpha^2$, $a, b, c \in Q$, for $3 \leq i \leq 6$. Set $\beta = \alpha^2$. Find a cubic in $Q[x]$ that has $\beta$ as a root.

4. Here we examine the polynomial $p(x) = x^4 + 1$. Let $\alpha, \beta, \gamma, \delta$ denote the complex roots of $p(x) = 0$.

(a) Find $\alpha, \beta, \gamma, \delta$ both in terms of polar coordinates $\alpha = re^{i\theta}, \ldots$ (this is actually easier for this particular problem) and in the Cartesian $\alpha = a + bi, \ldots$ forms and mark them on the complex plane.

(b) Give the factorization of $p(x)$ into irreducibles in $C[x]$.

(c) Give the factorization of $p(x)$ into irreducibles in $Re[x]$.

(d) Give the factorization of $p(x)$ into irreducibles in $(Q(\sqrt{2}))[x]$.

(e) Give the factorization of $p(x)$ into irreducibles in $(Q(i\sqrt{2}))[x]$. 
(f) Show \( p(x) \) is irreducible in \( \mathbb{Q}[x] \) using the following idea: If, say, \( p(x) = f(x)g(x) \), then, as \( p(x) \) factors into four linear factors in the first part above, \( f(x) \) and \( g(x) \) must be a product of some (but not all) of those factors. Try all possibilities for products of the linear factors (there aren’t that many) and check that none of them give an \( f(x) \in \mathbb{Q}[x] \).

(g) Show \( p(x) \) is irreducible in \( \mathbb{Q}[x] \) using the following idea: First (the easy part) show \( p(x) \) has no root in \( \mathbb{Z} \). Now suppose \( p(x) = f(x)g(x) \) where \( f(x), g(x) \in \mathbb{Z}[x] \) are monic quadratics. Then \( f(i) \mid p(i) \) for \( i = 0, 1 \). Writing \( f(x) = x^2 + ax + b \) the values \( f(0), f(1) \) determine \( a, b \). Show that in each case that \( f(x) \) is not a divisor of \( p(x) \). (Some grunt work here. A quick way to show \( f(x) \) does not divide \( p(x) \) is to check \( x = \pm 2 \).)

5. Assume as a fact that \( x^5 + x^2 + 1 \) is irreducible in \( \mathbb{Z}_2[x] \). (It is!) Set \( F = \mathbb{Z}_2[x]/(x^5 + x^2 + 1) \). Set \( F^* = F - \{0\} \).

(a) How many elements are in \( F \). Describe the elements.

(b) How many elements are in \( F^* \).

(c) Considering \( F^* \) as a group under multiplication find the order of \( x \), that is, the minimal \( n \) such that \( x^n = 1 \). (One can do this by brute force but there is a quick way!)

(d) Find the remainder, in \( \mathbb{Z}_2[x] \), when \( x^{31000002} + 1 \) is divided by \( x^5 + x^2 + 1 \). (Again, there is a quick way!)

Nothing is more fruitful - all mathematicians know it - than those obscure analogies, those disturbing reflections of one theory in another; those furtive caresses, those inexplicable discords; nothing also gives more pleasure to the researcher. The day comes when the illusion dissolves; the yoked theories reveal their common source before disappearing. As the Gita teaches, one achieves knowledge and indifference at the same time.

André Weil
(Note: “indifference” is a controversial translation of the original Sanskrit, “detachment” is often used instead)