I cannot live without people. – Pope Francis

1. In this problem, assume (important!) that $G$ is Abelian. Set $H = \{g \in G : g^5 = e\}$. (Warning: Expressions such as $x^{1/5}$ are not well defined. Do not use them!)
   (a) Show $H$ is a subgroup of $G$. Point out where the assumption that $G$ was Abelian was used.
   (b) Show $H$ is a normal subgroup of $G$. (Easy! – If you see it.)
   (c) Assume further that $G$ is finite and that $H = \{e\}$. Show that the map $\phi : G \to G$ given by $\phi(g) = g^5$ is an automorphism. (Definition: An automorphism is an isomorphism from a group to itself.)

2. Let $G$ be any group and $g$ a fixed element of $G$. Define $\phi : G \to G$ by $\phi(x) = g^{-1}xg$.
   (a) Show that $\phi$ is a homomorphism.
   (b) Show that $\phi$ is an injection. To do this you have to show that the equation $\phi(x) = e$ has only the solution $x = e$.
   (c) Show that $\phi$ is a surjection. To do this you have to show, given any $y \in G$, that the equation $\phi(x) = y$ has a solution. [With these three you may conclude that $\phi$ is an automorphism as defined above.]

3. Find an isomorphism $\phi : (\mathbb{Z}_{12}, +) \to (\mathbb{Z}_{13}^\ast, \cdot)$. (Idea: $\phi(1)$ determines $\phi$, try various $\phi(1)$ until one works.)

4. Let $G$ be the positive reals under multiplication and let $H$ be numbers $2^i$ where $i \in \mathbb{Z}$.
   (a) Show $H$ is a subgroup of $G$.
   (b) Show $H$ is a Normal subgroup of $G$. (Easy! – If you see it.)
   (c) Give a natural representation of the factor group $G/H$. By this I mean that each element should be uniquely describable as $a\bar{n}$ where $a$ ranges over some natural set. (So, for example, you couldn’t have $3.1$ and $12.4$ as $\bar{3.1} = \bar{12.4}$.) Have the identity represented as $\bar{1}$.
(d) Find all elements \( \pi \in G/H \) whose cube (in \( G/H \)) is the identity.  
(This is a bit tricky. There are precisely three solutions!)

5. Just for fun: What’s purple and commutes?

6. Some questions about the order of an element.

   (a) Let \( g \in G \) with \( o(g) = 100 \). For what \( i \) is \( g^i = e \)?

   (b) Let \( g \in G \) with \( g^{100} = e \). What are the possible values of \( o(g) \)?

   (c) Let \( \phi : G \rightarrow H \) be a homomorphism and let \( g \in G \) and set \( h = \phi(g) \). Suppose \( o(h) = 100 \). Assume \( g \) has finite order. What are the possible values of \( o(g) \)?

The world can be divided into those who love New York City and those who don’t. Those who love New York tend to be unusually lively people. They have to be. Characteristically, they are ambitious, curious, intellectually vigorous, culturally alive. Such people give New York City institutions great dynamism and some eccentricity.

James Hester 1924-2015, NYU President