The universe is not only queerer than we suppose but queerer than we can suppose.
– J.B.S. Haldane

1. Let $\alpha \in C$ be a root of $x^3 + x + 3$. (This cubic has no special properties.) Write $\alpha^i$ in the form $a + b\alpha + c\alpha^2$, $a, b, c \in Q$, for $3 \leq i \leq 6$. Set $\beta = \alpha^2$. Find a cubic in $Q[x]$ that has $\beta$ as a root.

2. Let $p(x) \in Z[x]$ be a monic polynomial of degree $n$, irreducible over $Q$. Write $p(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_0$. Let $\alpha \in C$ be a root of the equation $p(x) = 0$. Set

$$R = \{a_0 + a_1\alpha + \ldots + a_{n-1}\alpha^{n-1} : a_0, \ldots, a_{n-1} \in Z\}$$

Show that $R$ is a ring. (The hard part is closure under multiplication.) Now suppose further that $p(x)$ has constant term $b_0 = \pm 1$. Show that $\alpha^{-1} \in R$.

3. Here we examine the polynomial $p(x) = x^4 + 1$. Let $\alpha, \beta, \gamma, \delta$ denote the complex roots of $p(x) = 0$.

(a) Find $\alpha, \beta, \gamma, \delta$ both in terms of polar coordinates $\alpha = re^{i\theta}, \ldots$ (this is actually easier for this particular problem) and in the Cartesian $\alpha = a + bi, \ldots$ forms and mark them on the complex plane.

(b) Give the factorization of $p(x)$ into irreducibles in $C[x]$.

(c) Give the factorization of $p(x)$ into irreducibles in $Re[x]$.

(d) Give the factorization of $p(x)$ into irreducibles in $(Q(\sqrt{2})[x]$.

(e) Give the factorization of $p(x)$ into irreducibles in $(Q(i\sqrt{2})[x]$.

(f) Show $p(x)$ is irreducible in $Q[x]$ using the following idea: If, say, $p(x) = f(x)g(x)$, then, as $p(x)$ factors into four linear factors in the first part above, $f(x)$ and $g(x)$ must be a product of some (but not all) of those factors. Try all possibilities for products of the linear factors (there aren’t that many) and check that none of them give an $f(x) \in Q[x]$. 
(g) There are many ways to show that a polynomial is irreducible over 
$Q[x]$. Show $p(x)$ is irreducible over $Q[x]$ by some other method – 
your choice!

4. Let $f(z) = 10 + (i + 1)z^4 + 6z^5$. (Here $i = \sqrt{-1}$.) Find an angle 
$\phi$ (following the proof of the Fundamental Theorem of Algebra) such 
that $|f(10^{-100}e^{i\phi})| < 10 - 1.4 \cdot 10^{-400}$.

5. Let $\alpha \in C$ satisfy $\alpha^3 + \beta \alpha^2 + \gamma \alpha + \delta = 0$ where $\beta, \gamma, \delta$ satisfy cubics with 
coefficients in $Q$. (That is, $\beta^3 + a\beta^2 + b\beta + c = 0$ for some $a, b, c \in Q$, and 
similarly for $\gamma, \delta$.) (Possibly smaller degree polynomials are satisfied 
by these numbers.)

(a) Give an upper bound for the degree of $\alpha$ over $Q$.
(b) Show that the minimal polynomial of $\alpha$ over $Q$ does not have 
degree precisely five.

6. If $\alpha, \beta \in K$ are algebraic over $F$ of degrees $m$ and $n$ respectively and 
if $m, n$ are relatively prime, prove that $F(\alpha, \beta)$ is of degree 
$mn$ over $F$.

7. Let $p$ be an odd prime. Assume as a (true!) fact that when $\alpha, \beta \in Z_p^*$ 
are not squares then $\alpha/\beta$ is a square. Let $L, K$ be fields, both of size 
precisely $p^2$. Using the results on extension of degree two show that 
$L, K$ are isomorphic. ($L, K$ are isomorphic if there exists a bijection 
$\psi : L \rightarrow K$ that preserves addition and multiplication.)

Nothing is more fruitful - all mathematicians know it - than 
those obscure analogies, those disturbing reflections of one theory 
in another; those furtive caresses, those inexplicable discords; 
nothing also gives more pleasure to the researcher. The day 
comes when the illusion dissolves; the yoked theories reveal their 
common source before disappearing. As the Gita teaches, one 
achieves knowledge and indifference at the same time.

André Weil
(Note: “indifference” is a controversial translation of the original 
Sanskrit, “detachment” is often used instead)