Honors Algebra II V63.0349
Assignment 2
Due, Friday, Feb 5 in Recitation

Many persons who have not studied mathematics confuse it with arithmetic and consider it a dry and arid science. Actually, however, this science requires great fantasy.
– Sophia Kovalevsky

1. In \( \mathbb{Z}_7[x] \) let \( f(x) = 2x^5 + 3x^4 + 4x + 4 \) and \( g(x) = 3x^2 + x + 5 \). Find \( q(x), r(x) \) with \( f(x) = q(x)g(x) + r(x) \) and either \( r(x) = 0 \) or \( r(x) \) having smaller degree than \( g(x) \).

2. (*) What is the remainder when \( x^{1000000} \) is divided by \( x^3 + x + 1 \) in \( \mathbb{Z}_2[x] \). (There is a pattern!)

3. Further problems on \( \mathbb{Z}[\omega] \), as in assignment 1.
   (a) What are the possible values of \( |\alpha|^2, \alpha \in \mathbb{Z}[\omega] \), with \( |\alpha|^2 \leq 11 \). (Notation: For \( \alpha = a + bi \in \mathbb{C} \), \( |\alpha| = \sqrt{a^2 + b^2} \), the distance from \( \alpha \) to the origin on the complex plane.)
   (b) Factor the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 into irreducibles in \( \mathbb{Z}[\omega] \). Prove that you do indeed have irreducibles. [One aid: If \( \alpha = \beta \gamma \) then \( |\alpha|^2 = |\beta|^2 |\gamma|^2 \). Also, \( |\alpha|^2 = 1 \) exactly for the six units of assignment 1.]
   (c) Find the minimal positive real \( x \) with the following property: For any \( \beta \in \mathbb{C} \) there exists \( \alpha = a + b\omega \in \mathbb{Z}[\omega] \) with \( |\beta - \alpha| \leq x \). (The geometry of \( \mathbb{Z}[\omega] \) is particularly helpful here.)
   (d) Use the above and following the argument for \( \mathbb{Z}[i] \), prove that \( \mathbb{Z}[\omega] \) is a Euclidean Ring under \( d(\alpha) = |\alpha|^2 \).

4. Even more problems on \( \mathbb{Z}[\omega] \), as in assignment 1.
   (a) In the picture of \( \mathbb{Z}[\omega] \) (you can use the picture from the solutions to assignment one) mark (with a little circle) those points which are in the ideal (2).
   (b) Describe \( \mathbb{Z}[\omega]/(2) \) as \( \overline{\alpha_1}, \ldots, \overline{\alpha_r} \) for some specific \( \alpha_1, \ldots, \alpha_r \). (You have to figure out what \( r \) is!)
   (c) Give the multiplication table for \( \mathbb{Z}[\omega]/(2) \). Is it a field? (It is automatically a ring so to be a field every element has to have a multiplicative inverse.)
5. Let’s call an integral Domain $D$ together with a function $d : D - \{0\} \rightarrow \{0, 1, 2, \ldots\}$ a Banana Domain (not its real name!) if

(a) $d(\alpha) \leq d(\alpha\beta)$ for all nonzero $\alpha, \beta \in D$

(b) If $\alpha, \beta \in D - \{0\}$ and if there does not exist $q \in D$ with $\alpha = q\beta$ then there exist $a, b \in D$ with $a\alpha + b\beta \neq 0$ and (critically!) $d(a\alpha + b\beta) < d(\beta)$.

Prove that a Banana Domain is a P.I.D. (Hint: Follow the argument that a Euclidean Domain is a P.I.D.)

Math is natural. Nobody could have invented the mathematical universe. It was there, waiting to be discovered, and its crazy; its bizarre. – John Conway