I wasn’t sure anymore and I will tell you, it is a strange process to feel one’s mind changing, allowing ideas into your brain which it had once considered unthinkable. I cannot say it’s painful, or particularly pleasurable, but that it requires a certain relaxation of the hold one keeps over oneself, and is to that degree both a thrill and a horror.

– from The Chess Garden, by Brooks Hansen

The **FINAL EXAM** will be Monday, May 15, 10:00 a.m. - 11:50 a.m. in our regular classroom. Special office hours and further information on the exam will be placed on the website as the date gets closer.

With our final assignment we prove one of the most important results in all of mathematics: Any polynomial $f(x) \in \mathbb{C}[x]$ has a root. This is often called the Fundamental Theorem of Algebra. There are several proofs using methods from a variety of areas of mathematics. Here, naturally, we give a proof using Galois Theory. We let $\mathbb{R}, \mathbb{C}$ denote the reals and complex numbers. We shall use the following results:

1. Any $f(x)\mathbb{R}[x]$ of odd degree has a real root.

2. Every $\alpha \in \mathbb{C}$ has a square root.

3. [Sylow Theorem] If a group $G$ has $n$ elements and $q|n$ and $q$ is a prime power then there is a subgroup $H \subset G$ with precisely $q$ elements.

The root for odd degree real polynomials follows from the Mean Value Theorem. This is the only place in the argument where we use Analysis. All else is from Algebra.

1. Let $f(x) \in \mathbb{R}[x]$ have degree $n$ with no real roots. Let $\Omega$ be a splitting field for $f(x)$. (Note: We cannot assume $\Omega \subset \mathbb{C}$ since we are aiming to prove the Fundamental Theorem!) Let $G = \Gamma[\Omega : \mathbb{R}]$. Assume that $[\Omega : \mathbb{R}]$ is not a power of two. Write $[\Omega : \mathbb{R}] = 2^u s$ with $u \geq 0$ and $s \geq 3$ odd. Let (by (3)) $H \subset G$ with $2^u$ elements. Find $[H : \mathbb{R}]$. Use (1) to get a contradiction.

2. Show that if $\mathbb{R} \subset K$ and $[K : \mathbb{R}] = 2$ then $K \cong \mathbb{C}$. 
3. Now assume $[\Omega : \mathbb{R}] = 2^u$ with $u \geq 2$. Use (3) to find a tower

$$\mathbb{R} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_u = \Omega$$

with all $[K_{i+1} : K_i] = 2$.

4. Continuing, looking at $K_1, K_2$ above derive a contradiction.

5. Prove that the only finite extensions of $\mathbb{R}$ are $\mathbb{R}$ itself and $\mathbb{C}$.

6. Prove that the only finite extension of $\mathbb{C}$ is $\mathbb{C}$ itself.

7. Prove the Fundamental Theorem of Algebra!

The more I work, the more I see things differently, that is, everything gains in grandeur every day, becomes more and more unknown, more and more beautiful. The closer I come the grander it is, the more remote it is.

– Alberto Giacometti