Honors Algebra I
Assignment 12
Not to be Submitted Happy Thanksgiving

I was 21 years when I wrote this song
I’m 22 now, but I won’t be for long
Time hurries on
And the leaves that are green turn to brown
– Paul Simon, Leaves That are Green

These problems concern the Gaussian Integers $Z[i]$ with $d(\alpha) = \alpha \overline{\alpha} = a^2 + b^2$ where $\alpha = a + bi$ and $\overline{\alpha}$ represents the complex conjugate $a - bi$. By prime we mean in the integers $Z$ but by Gaussian prime we mean in $Z[i]$. For example, 5 is a prime but not a Gaussian prime as $5 = (2 + i)(2 - i)$. When we say “factors into $s$ primes” below, we count with repetition, so that $12 = 2 \cdot 2 \cdot 3$ factors into 3 primes. We write $x \sim y$ if $x = yu$ with $u$ a unit.

1. Show that $\gamma$ is a unit if and only if $\overline{\gamma}$ is a unit.

2. Show that $\alpha$ is a Gaussian prime if and only if $\overline{\alpha}$ is a Gaussian prime.

3. There is precisely one Gaussian prime $\alpha$ such that $\alpha \sim \overline{\alpha}$, other that those which are $\sim p$ for some integer $p$. Which $\alpha$ is it, and why is it the only one?

4. Show that $\alpha$ factors into $t$ Gaussian primes if and only if $\overline{\alpha}$ factors into $t$ Gaussian primes.

5. Show that if $p$ is a prime then either it is a Gaussian Prime or $p = \alpha \overline{\alpha}$ where $\alpha, \overline{\alpha}$ are Gaussian Primes.

6. Show that if $d(\alpha)$ is a prime then $\alpha$ is a Gaussian prime.

7. Show that if $n$ factors into $s$ primes then it factors into at least $s$ Gaussian primes.

8. Show that if $d(\alpha)$ factors into three or more primes then $\alpha$ cannot be a Gaussian prime. (Idea: look at the factorization of $d(\alpha) = \alpha \overline{\alpha}$.)

9. Suppose $d(\alpha) = pq$ where $p, q$ are distinct primes. Show that $\alpha$ cannot be a Gaussian prime.
10. Suppose $p$ is a prime and there is no expression $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$. Show that $p$ is a Gaussian prime.

11. Suppose $p$ is a prime and there is an expression $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$. Show that $p$ is not a Gaussian prime.

12. Show that if $d(\alpha) = p^2$ and $\alpha$ is a Gaussian prime then it must be that $\alpha \sim p$ and there is no expression $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$.

13. Give the prime factorizations 5, 13, 17 in $\mathbb{Z}[i]$. Set $n = 1105 = 5 \cdot 13 \cdot 17$. Give the prime factorization of $n$ in $\mathbb{Z}[i]$. Use this to find four “distinct” ways to write $n = \beta \overline{\beta}$. (That is, switching the order or multiplying by units gives the “same” way.) Use this to find four expressions of $n$ as the sum of two squares.

14. Generalizing: Suppose $n$ is the product of $s$ distinct odd primes $p_1, \ldots, p_s$, each of which is not a Gaussian Prime. Show that $n$ can be written as the sum of two squares in $2^{s-1}$ different ways.

Homeward bound
I wish I was
Homeward bound
Home, where my thoughts escaping
Home, where my musics playing
Home, where my love lies waiting
Silently for me
– Paul Simon, *Homeward Bound*