I doubt sometimes whether a quiet and unagitated life would have suited me – yet I sometimes long for it. – Byron

1. Let $K : Q$ be a normal extension, set $G = \Gamma[K : Q]$. Let $H$ be a subgroup of $G$. Let $a = |H|$, $s = |G|$. Let $\alpha \in K$. Set $\gamma = \sum_{\sigma \in H} \sigma(\alpha)$.

(a) Show that $\tau(\gamma) = \gamma$ for all $\tau \in H$.
(b) Show that $H \subseteq Q(\gamma)^*$.
(c) Deduce (using the Galois Correspondence Theorem) an upper bound on $[Q(\gamma) : Q]$.

2. Let $K : Q$ be a normal extension and assume that $K$ is not a subset of the reals.

(a) Show that $K$ is closed under complex conjugation. That is, let $\alpha = a + bi \in K$ with $a, b$ real. Show that $a - bi \in K$. (Warning: This is not true for any field extension, you must use that $K : Q$ is normal.
(b) Let $a + bi \in K$ with $a, b$ real. Show that $a \in K$ and $bi \in K$.
(c) Let $L$ be the field of real numbers $a \in K$. Prove that $[K : L] = 2$.

3. Let $G = (Z_4 \times Z_6, +)$

(a) Find two subgroups $H \subset G$ with precisely 12 elements.
(b) Find a third subgroup $H \subset G$ with precisely 12 elements.
(c) (*) Prove there aren’t any more $H$.
(d) Let $K : Q$ be normal with $\Gamma(K : Q)$ isomorphic to $G$ above. Show (using the Galois Correspondence Theorem) that $K$ has precisely three distinct nontrivial square roots. (We count $\sqrt{a}$ and $\sqrt{q^2a}$ as the same.)
(e) Let $K = Q(\epsilon)$ with $\epsilon = e^{2\pi i/35}$. Find an injective homomorphism from $\Gamma[K : Q]$ to $G$ as above. (Thus $\Gamma[K : Q]$ is isomorphic to a subgroup of $G$. Actually $\Gamma[K : Q] \cong G$ and we will show that later.)

I cannot live without people. – Pope Francis