Honors Algebra, Assignment 10
Due Friday, November 13

The truth may be puzzling. It may take some work to grapple with. It may be counterintuitive. It may contradict deeply held prejudices. It may not be consonant with what we desperately want to be true. But our preferences do not determine what’s true. We have a method, and that method helps us to reach not absolute truth, only asymptotic approaches to the truth; never there, just closer and closer, always finding vast new oceans of undiscovered possibilities.
– Carl Sagan

1. Let \( R \) be a ring and \( M \subset R \) an ideal. Assume \( M \neq R \) (but do not assume \( M \) is maximal). Let \( a \in R \) with \( a \notin M \).

   (a) Assume there exists an ideal \( N \) with \( M \subset N \subset R \) and \( N \neq M, R \) and \( a \in N \). Prove that \( a \) has no multiplicative inverse in \( R/M \).

   (b) Assume there does not exist an ideal \( N \) with \( M \subset N \subset R \) and \( N \neq M, R \) and \( a \in N \). Prove that \( a \) has a multiplicative inverse in \( R/M \).

2. Let \( R \) be a ring and let \( a, b \in R \). Set

\[
M = \{ ar + bs : r, s \in R \}
\]

Prove that \( M \) is an ideal. **Notation:** We will write \( M = (a, b) \).

3. Let \( Z[i] = \{ a + bi : a, b \in Z \} \) where \( i = \sqrt{-1} \), the usual Gaussian Integers.

   (a) For \( \alpha \in Z[i] \) define (this is called a **norm**) \( N(\alpha) = |\alpha|^2 \), where \( | \cdot | \) is the usual complex number absolute value, that is \( |c+di| = \sqrt{c^2 + d^2} \). Give a formula for \( N(\alpha) \) for \( \alpha \in Z[i] \). Show \( N(\alpha \beta) = N(\alpha)N(\beta) \).

   (b) Precisely which elements of \( Z[i] \) have multiplicative inverses? (Use the norm to show that you have everything.)

   (c) Define \( \phi : Z[i] \to Z[i] \) by \( \phi(a + bi) = a - bi \). (This is generally known as **complex conjugation**.) Show that \( \phi \) is a homomorphism by showing \( \phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta) \) and \( \phi(\alpha \beta) = \phi(\alpha)\phi(\beta) \). Show that \( \phi \) has kernel \( \{0\} \).
(d) A number $\alpha \in \mathbb{Z}[i]$ is called *composite* if we can write $\alpha = \beta \gamma$ where neither $\beta$ nor $\gamma$ have multiplicative inverses. (That last condition is to avoid “trivial” factorizations like $23 = i(-23i)$.) If it is nonzero, not a unit, and not composite it is called *prime*. Show that $2$ is composite. Show that $41$ is composite. Show $7 + 2i$ is prime. (Idea: Use the norm)

4. Set 

$$R = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$$

(a) Give a formula for $N(\alpha)$ (as defined above) for $\alpha = a + b\sqrt{-5} \in R$.

(b) Precisely which elements of $R$ have multiplicative inverses? (Use the norm to argue that you have all of them.)

(c) Set 

$$I = \{2\alpha + (1 + \sqrt{-5})\beta : \alpha, \beta \in R\}$$

Plot those $(a, b)$ with $-4 \leq a, b \leq +4$ so that $a + b\sqrt{-5} \in I$.

(d) Show that $I$ is an ideal.

(e) Show that $1 \not\in I$.

(f) Show that $I$ is *not* a principal ideal. (Idea: Assume $I = (\kappa)$ and use the properties of $N(\cdot)$ above.)

(g) Find representatives of $R/I$. What well known field is it isomorphic to?

My father took his stick and began writing an equation in the sand. Like all the rest of them, this one was busy with $x$’s and $y$’s resting on top of one another on dash-shaped bunks. Letters were multiplied by symbols, crowded into parentheses, and set upon by dwarfish numbers drawn at odd angles.

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