Honors Algebra II
Assignment 10
Due Friday, April 14 in recitation

She guessed at it all, what might live, moving purposefully or drifting aimlessly, under the deep water around her, but she didn’t think too much about any of it. It was enough to be aware of the million permutations possible around her, and take comfort in knowing she would not, and really could not, know much at all.
Dave Eggers, The Circle (movie coming out soon!)

1. Let $F \subset L \subset K$ be fields in $C$. Assume that $K : F$ is normal and that $L : F$ is normal. Let $\tau \in \Gamma[K : F]$ and $\sigma \in \Gamma[K : L]$. Let $l \in L$

(a) Argue that $\tau(l) \in L$.

(b) Show that $(\tau \sigma \tau^{-1})(l) = l$.

(c) From the above show that $\Gamma[K : L]$ is a normal subgroup (get out those Algebra I notes!) of $\Gamma[K : F]$. (Assume it's already been shown that it is a subgroup. You only need show the normal part.)

(d) In the case $F = Q, L = Q(\omega), K = F(\alpha, \omega)$ (with $\alpha = 2^{1/3}, \omega = e^{2\pi i/3}$ as in our standard example) give the groups $\Gamma[K : L]$ and $\Gamma[K : F]$ explicitly in terms of permutations of $\alpha, \beta = \alpha \omega, \gamma = \alpha \omega^2$.

2. Let $F \subset K$ be fields in $C$ with $[K : F] = 2$. Prove that $K : F$ is a normal extension.

3. Let $K_1, K_2$ be normal extensions of $Q$. Let $M$ denote the minimal field containing $K_1 \cup K_2$. Prove that $M$ is a normal extension of $Q$. (One approach: Write $K_1 = Q(\alpha_1, \ldots, \alpha_r)$ where the $\alpha_i$ are all the roots of some $p(x) \in Q[x]$ and similarly write $K_2 = Q(\beta_1, \ldots, \beta_s)$.

4. Let $\alpha$ be a root of $f(x) = x^3 + x^2 - 2x - 1 \in Q[x]$.

(a) Show $f(x)$ is irreducible over $Q$. Note: You should assume this in what follows.

(b) Find $[Q(\alpha) : Q]$.

(c) Set $\beta = -1/(\alpha + 1)$. Find $\beta$ in the form $a + b\alpha + c\alpha^2$. 
(d) Show that $f(\beta) = 0$. (Bit of grunt work here!)

(e) Find $\gamma \in Q(\alpha)$, $\gamma, \beta, \alpha$ distinct, with $f(\gamma) = 0$. (Idea: If $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ then $\alpha + \beta + \gamma$ is determined.)

(f) Deduce that $Q(\alpha) : Q$ is normal.

(g) Argue that $\Gamma(Q(\alpha) : Q)$ has precisely three elements. Which permutations of $(\alpha, \beta, \gamma)$ do the three automorphisms correspond to? What well known group is $\Gamma(Q(\alpha) : Q)$ isomorphic to? (Hint: There is only one group on three elements!)

(h) Let $K$ be the fixed field of $\Gamma(Q(\alpha) : Q)$. (That is, $K$ is those $\kappa \in K$ such that $\sigma(\kappa) = \kappa$ for all $\sigma \in \Gamma(Q(\alpha) : Q)$.) Prove that $K = Q$.

5. As in last week’s assignment set $\alpha = 2^{1/4}$, $\beta = i\alpha$, $\gamma = -\alpha$, $\delta = -i\alpha$. Set $p(x) = x^4 - 2$. Set $K = Q(\alpha, \beta, \gamma, \delta)$. Set $L = Q(i)$. Also, set $M = Q(\alpha)$ and $N = Q(\sqrt{2})$.

(a) Give the factorization of $p(x)$ into irreducible factors in $Q[x]$.

(b) Give the factorization of $p(x)$ into irreducible factors in $K[x]$.

(c) Give the factorization of $p(x)$ into irreducible factors in $M[x]$.

(d) Give the factorization of $p(x)$ into irreducible factors in $N[x]$.

He rarely copied box scores into the Book, but today it seemed the right thing to do. All those zeroes! He decided for zeroes he’d use red ink. Zero: the absence of number, an incredible idea! Only infinity compared to it, and no batter could hit an infinite number of home runs - no, in a way, the pitchers had it better. Perfection was available to them.


– Robert Coover