Honors Algebra II V63.0349
Assignment 1
NOT TO BE SUBMITTED

I have no home. The world is my home. – Paul Erdős

Prof. Spencer’s email: spencer@cims.nyu.edu. For despamification, please make subject matter Algebra.

Special Note: There is no recitation the first week,

1. An element $u \in R$ ($R$ a ring) is called a \textit{unit} if it has a multiplicative inverse, that is, there exists an element $v \in R$ such that $uv = 1$. (When this occurs we can write $v = u^{-1}$ or $v = \frac{1}{u}$.) Let $U$ be the set of units of $R$. Show that $U$ forms a group under multiplication. The template for this is:

(a) \textbf{Identity:} $1 \in U$

(b) \textbf{Inverse:} If $u \in U$ then $u^{-1} \in U$.

(c) \textbf{Product:} If $u, v \in U$ then $uv \in U$.

2. Define (this will be standard)

\[ Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\} \]

Show that $Z[\sqrt{2}]$ is a Ring. The template for showing $R$ is a ring when $R$ is a subset of the complex numbers is:

(a) \textbf{Identity:} $0, 1 \in R$

(b) \textbf{Inverse:} If $u \in R$ then $-u \in R$.

(c) \textbf{Sum:} If $u, v \in R$ then $u + v \in R$.

(d) \textbf{Product:} If $u, v \in R$ then $uv \in R$.

3. Let $\alpha = a + b\sqrt{2} \in Z[\sqrt{2}]$. Show

(a) If $a^2 - 2b^2 = \pm 1$ then $\alpha$ is a unit.

(b) (*) \textbf{[The asterisk indicates a more difficult, but still assigned, problem.]} If $\alpha$ is a unit then $a^2 - 2b^2 = \pm 1$.

4. Define (this will be standard)

\[ Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \]

Prove that $Q(\sqrt{2})$ is a field. The template for showing $F$ is a field when $F$ is a subset of the complex numbers is:
(a) Identity: 0, 1 ∈ F
(b) Inverse: If u ∈ F then −u ∈ F.
(c) Sum: If u, v ∈ F then u + v ∈ F.
(d) Product: If u, v ∈ F then uv ∈ F.
(e) Multiplicative Inverse: If u ∈ F and u ≠ 0 then u⁻¹ ∈ F

Note that the first four properties are those for a ring so if you already know that F is a ring you can skip them.

5. Set
\[ \omega = e^{2\pi i / 3} = \frac{-1 + i\sqrt{3}}{2} \]
and set
\[ Z[\omega] = \{ a + b\omega : a, b \in \mathbb{Z} \} \]
[We use the notation \( e^{i\theta} = \cos \theta + i \sin \theta \). Any nonzero complex \( \alpha \) may be uniquely written \( \alpha = re^{i\theta} \) with \( r > 0 \) real and \( 0 \leq \theta < 2\pi \) and has polar coordinates \((r, \theta)\) when placed on the complex plane.]

(a) Draw a careful picture marking the points of \( Z[\omega] \) on the complex plane. You should get a pleasing pattern.
(b) Show \( Z[\omega] \) is a ring. (Product is the hard part!)
(c) (*) Find all units (with proof that you have all!) of \( Z[\omega] \).

What a chimera then is man! What a novelty, what a monster, what a chaos, what a contradiction, what a prodigy! Judge of all things, feeble earthworm, repository of truth, sewer of uncertainty and error, the glory and the scum of the universe.
– Blaise Pascal