I have no home. The world is my home. – Paul Erdős

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1. An element $u \in R$ ($R$ a ring) is called a unit if it has a multiplicative inverse, that is, there exists an element $v \in R$ such that $uv = 1$. (When this occurs we can write $v = u^{-1}$ or $v = \frac{1}{u}$.) Let $U$ be the set of units of $R$. Show that $U$ forms a group under multiplication. The template for this is:

   (a) Identity: $1 \in U$
   (b) Inverse: If $u \in U$ then $u^{-1} \in U$.
   (c) Product: If $u, v \in U$ then $uv \in U$.

2. Define (this will be standard)

   \[ Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\} \]

   Show that $Z[\sqrt{2}]$ is a Ring. The template for showing $R$ is a ring when $R$ is a subset of the complex numbers is:

   (a) Identity: $0, 1 \in R$
   (b) Inverse: If $u \in R$ then $-u \in R$.
   (c) Sum: If $u, v \in R$ then $u + v \in R$.
   (d) Product: If $u, v \in R$ then $uv \in R$.

3. Let $\alpha = a + b\sqrt{2} \in Z[\sqrt{2}]$. Show

   (a) If $a^2 - 2b^2 = \pm 1$ then $\alpha$ is a unit.
   (b) (*) [The asterisk indicates a more difficult, but still assigned, problem.] If $\alpha$ is a unit then $a^2 - 2b^2 = \pm 1$.

4. Define (this will be standard)

   \[ Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \]

   Prove that $Q(\sqrt{2})$ is a field. The template for showing $F$ is a field when $F$ is a subset of the complex numbers is:
(a) **Identity:** \(0, 1 \in F\)
(b) **Inverse:** If \(u \in F\) then \(-u \in F\).
(c) **Sum:** If \(u, v \in F\) then \(u + v \in F\).
(d) **Product:** If \(u, v \in F\) then \(uv \in F\).
(e) **Multiplicative Inverse:** If \(u \in F\) and \(u \neq 0\) then \(u^{-1} \in F\)

Note that the first four properties are those for a ring so if you already know that \(F\) is a ring you can skip them.

5. Set

\[ \omega = e^{2\pi i/3} = \frac{-1 + i\sqrt{3}}{2} \]

and set

\[ Z[\omega] = \{a + b\omega : a, b \in Z\} \]

[We use the notation \(e^{i\theta} = \cos \theta + i\sin \theta\). Any nonzero complex \(\alpha\) may be uniquely written \(\alpha = re^{i\theta}\) with \(r > 0\) real and \(0 \leq \theta < 2\pi\) and has polar coordinates \((r, \theta)\) when placed on the complex plane.]

(a) Draw a **careful** picture marking the points of \(Z[\omega]\) on the complex plane. You should get a pleasing pattern.
(b) Show \(Z[\omega]\) is a ring. (**Product** is the hard part!)
(c) (*) Find all units (with proof that you have all!) of \(Z[\omega]\).

What a chimera then is man! What a novelty, what a monster, what a chaos, what a contradiction, what a prodigy! Judge of all things, feeble earthworm, repository of truth, sewer of uncertainty and error, the glory and the scum of the universe.
– Blaise Pascal