I like things that look difficult and intractable to solve – they challenge me because they are more interesting to figure out.
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Problems marked with (*) are more difficult.
Problems marked with (-) are less difficult.
Do all problems. Maximal score 165.

Notations: $\mathbb{R}$ is the reals, $\mathbb{Z}$ the integers, $\mathbb{Q}$ the rational numbers. $S_n$ is the set of permutations of $1, \ldots, n$. $A_n$ is the set of even permutations of $1, \ldots, n$. $\mathbb{Z}[i]$ is the Gaussian Integers.

1. (5) (-) Give an explicit ideal $N$ in the Gaussian Integers $\mathbb{Z}[i]$ such that $(13) \subset N \subset \mathbb{Z}[i]$ and $(13) \neq N$ and $N \neq \mathbb{Z}[i]$. (No proof required.)

2. (15) (*) Let $G = \mathbb{R}$ and $H = \mathbb{Q}$, both under addition. Prove that, other than the identity, no element of $G/H$ has finite order.

3. (20) Let $\sigma \in S_n$. Let $\tau = \gamma_1 \cdots \gamma_r$ where each $\gamma_j$ is a three cycle. (That is, $\gamma_j = (a_jb_jc_j)$ for some distinct $a_j, b_j, c_j$.) Assume $\sigma(i) = \tau(i)$ for $1 \leq i \leq n - 2$.

   (a) (5) Show that if $\sigma \in A_n$ then $\sigma = \tau$.
   (b) (5) (-) Show that if $\sigma = \tau$ then $\sigma \in A_n$.
   (c) (10) With $n = 7$ and $\sigma = (1462375)$ find explicit $\gamma_1, \ldots, \gamma_r$ (you decide what $r$ is) so that $\sigma = \gamma_1 \cdots \gamma_r$.

4. (15) Let $R$ be a ring and $M \subset R$ a maximal ideal. Prove that $R/M$ is a field. (You can assume $R/M$ is a ring without proof.)

5. (10) Let $p$ be a prime in $\mathbb{Z}$.

   (a) (5) (-) Prove that if there exist integers $x, y$ with $p = x^2 + y^2$ then $p$ is not a Gaussian prime.
   (b) (5) Prove that if $p$ is not a Gaussian prime then there exist integers $x, y$ with $p = x^2 + y^2$.

6. (5) (-) Give an explicit unit $\beta \in \mathbb{Z}[\sqrt{101}]$ with $\beta \neq \pm 1$. 
7. **Important:** You are not allowed to use the Fundamental Theorem of Abelian Groups for this problem. You are allowed (and encouraged) to assume earlier parts when doing later parts of the problem.

Let $G$ be an Abelian group under addition with $p^3$ elements, $p$ prime. Assume that no element $y \in G$ with $o(y) = p^3$ and there is an element of order $p^2$. Let $x \in G$ with $o(x) = p^2$. Let $H = \{ix : 0 \leq i < p^2\}$. For any $z \in G$ let $z = z + H \in G/H$. Let $y \in G$ with $y \notin H$.

(a) (5) (-) Prove that $o(y) = p$.
(b) (5) Prove that $py = ix$ for some $0 \leq i < p^2$.
(c) (5) Prove that $py = pwx$ for some $0 \leq w < p$.
(d) (10) Prove that there exists $z \in G$ with $y = z$ and $o(z) = p$.

8. (15) Let $G$ be an Abelian group. Let $x, y \in G$ with $o(x) = m$ and $o(y) = n$. Assume $m, n$ are relatively prime. Prove $o(xy) = mn$.

9. (45) Set (with, as usual, $i = \sqrt{-1}$)

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

and

$$Z[\omega] = \{a + b\omega : a, b \in Z\}$$

Set, for $\alpha \in Z[\omega], \alpha \neq 0$, set $d(\alpha) = |\alpha|^2$ with $|\cdot|$ the usual absolute value of a complex number.

(a) (5) Draw the points $x + iy \in Z[\omega]$ with $y = 0, \frac{1}{2}\sqrt{3}, \sqrt{3}$ on the complex plane so that a clear pattern emerges. ($\sqrt{3} \sim 1.73$)
(b) (5) Find explicit $a, b$ with $\omega^2 = a + b\omega$.
(c) (5) Find $d(a + b\omega)$ as a polynomial in $a, b$.
(d) (5) (*) Find all the units of $Z[\omega]$. (No proof required.)
(e) (5) Show $d(\alpha) \leq d(\alpha\beta)$ for all $\alpha, \beta \in Z[\omega] - \{0\}$.
(f) (10) Let $\alpha, \beta \in Z[\omega], \beta \neq 0$. Prove that there exist $q, r \in Z[\omega]$ with $\alpha = q\beta + r$ and either $r = 0$ or $d(r) < d(\beta)$. (Idea: Imitate argument for $Z[i]$.)
(g) (5) Factor 3 into primes in $Z[\omega]$, showing that each of the factors (warning: there may be only one!) is prime.
(h) (5) Factor $3 + \omega$ into primes in $Z[\omega]$, showing that each of the factors (warning: there may be only one!) is prime.

10. (10) Let $R$ be a ring, $M, N$ ideals in $R$. Set

$$W = \{m + n : m \in M, n \in N\}$$

Prove $W$ is an ideal.

These answers are not quite correct; but at least they are genuine, as the results of mental work only. “A poor thing, Sir, but mine own!”

From *Pillow Problems* by Charles Dodgson (aka Lewis Carroll)