Here we solve the general cubic equation

\[ f(x) = x^3 + ax^2 + bx + c = 0 \]  

(1)

The roots add to \(-\frac{a}{3}\). Adding \(\frac{a}{3}\) to each they add to zero. So we translate, setting \(x = y - \frac{a}{3}\). Now the \(y^2\) terms cancel and the new equation is

\[ g(y) = y^3 + dx + f = 0 \]  

(2)

The linear transformation \(y = z\alpha\) then replaces \(d\) by \(d\alpha^{-2}\). With foresight, we want this to be \(-3\). So set \(\alpha = \sqrt[3]{-d/3}\) so that the new equation is

\[ h(z) = z^3 - 3z + g = 0 \]  

(3)

Now comes the strange part. We make the substitution

\[ z = w + w^{-1} \]  

(4)

so that

\[ a(w) = (w + w^{-1})^3 - 3(w + w^{-1}) + g = w^3 + g + w^{-3} = 0 \]  

(5)

as (this being why we picked \(-3\)) the \(w\) and \(w^{-1}\) coefficients cancel. Thus we have a degree six (!) equation

\[ w^6 + gw^3 + 1 = 0 \]  

(6)

However, it is a very special degree six equation. Setting \(v = w^3\) we get a quadratic

\[ v^2 + gv + 1 = 0 \]  

(7)

for \(v\). We solve for \(v\) (using square roots) and then back to \(w\) (using cube roots). Then equation (4) gives \(z\), \(y = z\alpha\) gives \(y\) and finally \(x = y - \frac{a}{3}\) gives the desired \(x\).

Its actually quite a mess to write down. Its called Cardano’s Formula and was found in the sixteenth century.

**Remark:** It appears that we are getting six solutions as there are two choices for \(v\) and then, for each, three choices for \(w\). Its a nice exercise to show that these collapse leaving only three different solutions for \(z\).