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VARIATIONAL ANALYSIS OF THE ABSCISSA MAPPING FOR POLYNOMIALS*

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Abstract. The abscissa mapping on the affine variety $\mathcal{M}_n$ of monic polynomials of degree n is the mapping that takes a monic polynomial to the maximum of the real parts of its roots. This mapping plays a central role in the stability theory of matrices and dynamical systems. It is well known that the abscissa mapping is continuous on $\mathcal{M}_n$, but not Lipschitz continuous. Furthermore, its natural extension to the linear space $\mathcal{P}_n$ of polynomials of degree n or less is not continuous. In our analysis of the abscissa mapping, we use techniques of modern nonsmooth analysis described extensively in Variational Analysis (R. T. Rockafellar and R. J.-B. Wets, Springer-Verlag, Berlin, 1998). Using these tools, we completely characterize the subderivative and the subgradients of the abscissa mapping, and establish that the abscissa mapping is everywhere subdifferentially regular. This regularity permits the application of our results in a broad context through the use of standard chain rules for nonsmooth functions. Our approach is epigraphical, and our key result is that the epigraph of the abscissa map is everywhere Clarke regular.

Key words. nonsmooth analysis, polynomials, stability, subgradient, Clarke regular

AMS subject classifications. 34D05, 34D20, 49J52

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Let $\mathcal{P}_n$ denote the linear space of complex polynomials of degree n or less, and let $\mathcal{M}_n$ denote the affine variety in $\mathcal{P}_n$ consisting of the monic polynomials of degree n. In this article we study variational properties of the *abscissa mapping*

$$a : \mathcal{M}_n \to \mathbb{R}$$

given by

$$a(p) = \max \{ \operatorname{Re} \zeta \mid p(\zeta) = 0 \} .$$

Our study is partly motivated by the need to provide tools for understanding the variational behavior of the *spectral abscissa mapping* on the n by n complex matrices defined by

$$\alpha(M) = a(\det(\lambda I - M)).$$

Properties of the spectral abscissa are closely tied to stability theory for matrices and dynamical systems. Thus, the variational behavior of the spectral abscissa has important consequences for the sensitivity of the stability properties of such systems under perturbation. In [BO], we apply the variational results obtained in this paper to study the variational behavior of the spectral abscissa map.

The abscissa mapping has a number of characteristics that make it difficult to analyze. It is well known that a is continuous, but not Lipschitz continuous, on

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$\mathcal{M}_n$. In addition, the natural extension of a to all of $\mathcal{P}_n$ is not continuous at any point of the subspace $\mathcal{P}_{n-1}$. In this paper, we show that the techniques of modern nonsmooth analysis described in the recent book [RW98] are ideally suited to the study of mappings of this type. Thus, a secondary purpose of this paper is to illustrate the usefulness of the nonsmooth analysis techniques developed by many authors over the last 30 years by applying them to a classical function of great practical importance. Using techniques from nonsmooth analysis, we are able to establish that the abscissa mapping is everywhere *subdifferentially regular*. This remarkable result has major consequences for the development of a calculus for the variational behavior of the abscissa mapping under composition.

It needs to be stated that our analysis owes a great debt to earlier work of Levantovskii [Lev80]. Levantovskii studied the set of stable polynomials, i.e., the set of polynomials whose abscissa is nonpositive, and provided an outline for the derivation of the tangent cone to this set. We generalize this proof technique to establish the key result of section 1 (Theorem 1.2).

The paper is organized as follows; we assume that the reader is familiar with [RW98]. Section 1 is devoted to the derivation of the *subderivative* of a. This is done via an *epigraphical* approach, where we derive the formula for the subderivative from a description of the *tangent cone* to the epigraph of the abscissa mapping a. In addition, we develop some basic tools that relate the prime factorization of a polynomial to a factorization of the tangent cone. The key to this result is the local factorization Lemma 1.4. In section 2, we use the representation of the tangent cone obtained in section 1 to derive a representation for the set of *regular normals* to the epigraph of a. This in turn yields a representation for the set of *regular subgradients* for a at any point in $\mathcal{M}_n$. In section 3, we establish that the abscissa mapping is everywhere subdifferentially regular. The key result is that the epigraph of the abscissa map is Clarke regular.

Most of the notation that we use is introduced as it is required. However, it is useful to briefly describe our conventions for discussing polynomials in their distinct roles as points in the linear space $\mathcal{P}_n$ and as functions over the complex field. One could identify $\mathcal{P}_n$ with $\mathbb{C}^{n+1}$ and attempt to derive the variational properties of a as a mapping on $\mathbb{C}^{n+1}$, but this would completely ignore the very rich underlying algebraic structure of polynomials. Since it is the roots of polynomials that lie at the heart of the mapping a, it is the polynomial perspective that drives our analysis. Given a polynomial $p \in \mathcal{P}_n$, we will always use the Greek letter λ to denote the indeterminant associated with representing the polynomial as a function. Thus we write $p(\lambda)$ as the associated polynomial function. Monomials and shifted monomials play a central role in our analysis. For this reason we give them a special notation so that we can discuss them as points in $\mathcal{P}_n$. We write

$$e_{(\ell,\lambda_0)}(\lambda) = (\lambda - \lambda_0)^\ell.$$

1. The subderivative and the tangent cone. To apply the tools developed in [RW98], we first extend the definition of a to the entire linear space $\mathcal{P}_n$:

$$a : \mathcal{P}_n \to \mathbb{R}$$

is given by

$$a(p) = \begin{cases} \max\{\operatorname{Re}\zeta \mid p(\zeta) = 0\} & \text{if } p \in \mathcal{M}_n, \\ +\infty & \text{otherwise.} \end{cases}$$

This extension allows us to focus our attention on the set of monic polynomials. In particular, we have $\operatorname{dom}(a) = \{p \mid a(p) < +\infty\} = \mathcal{M}_n$. Given $p \in \mathcal{M}_n$, our goal is to derive a formula for $da(p)$, the *subderivative* of the mapping a. Following [RW98, Definition 8.1], the subderivative of a at a point $p \in \mathcal{M}_n$ is the mapping $da(p) : \mathcal{P}_n \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$da(p)(\hat{q}) = \liminf_{\substack{\tau \searrow 0 \\ q \to \hat{q}}} \frac{a(p + \tau q) - a(p)}{\tau},$$

where the parameter τ is understood to be real. Since a is $+\infty$ on $\mathcal{P}_n \backslash \mathcal{M}_n$, we have

$$\operatorname{dom}(da(p)) = \{p \mid da(p) < +\infty\} \subset \mathcal{P}_{n-1}.$$

Hence, we restrict our attention to the behavior of $da(p)$ on the subspace $\mathcal{P}_{n-1}$.

We approach the problem of computing $da(p)$ from an epigraphical perspective. The epigraph of a is the set

$$\operatorname{epi}(a) = \{(p, \mu) \mid a(p) \le \mu < +\infty\}.$$

Using this set, we can construct $da(p)$ from the formula

$$\operatorname{epi}(da(p)) = T_{\operatorname{epi}(a)}(p, a(p)) \tag{1.1}$$

[RW98, Theorem 8.2]. Here $T_{\operatorname{epi}(a)}(p, a(p))$ is the *tangent cone* to the set $\operatorname{epi}(a)$ at the point $(p, a(p))$. For a subset C of a finite dimensional linear space X, we have

$$T_C(x) = \left\{ d \;\middle|\; \begin{array}{c} \exists\, \{x^k\} \subset C \text{ and } \{t_k\} \subset \mathbb{R}_+ \text{ such that} \\ x^k \to x,\ t_k \searrow 0, \text{ and } t_k^{-1}(x^k - x) \to d \end{array} \right\} \tag{1.2}$$

$$= \left\{ \gamma d \;\middle|\; \begin{array}{c} \gamma \ge 0, \text{ and there exits } \{x^k\} \subset C \\ \text{with } x^k \to x \text{ such that } d = \lim\limits_{k\to\infty} \dfrac{x^k - x}{\|x^k - x\|} \end{array} \right\}, \tag{1.3}$$

where $\mathbb{R}_+$ is the set of nonnegative real numbers and $\|\cdot\|$ is any norm on X. By considering $\mathcal{P}_{n-1}$ as a subspace of $\mathcal{P}_n$, we have

$$T_{\operatorname{epi}(a)}(p, \mu) \subset \mathcal{P}_{n-1} \times \mathbb{R} \qquad \text{for all } \mu \ge a(p), \tag{1.4}$$

since a is $+\infty$ on $\mathcal{P}_n \backslash \mathcal{M}_n$. In particular,

$$T_{\operatorname{epi}(a)}(p, \mu) = \mathcal{P}_{n-1} \times \mathbb{R} \qquad \text{whenever } \mu > a(p), \tag{1.5}$$

since a is continuous on $\mathcal{M}_n$.

In our first lemma we show that the tangential geometry of $\operatorname{epi}(a)$ remains essentially unchanged under the linear transformations corresponding to a uniform shift of the roots. For each $\lambda_0 \in \mathbb{C}^n$ define the linear transformation $H_{\lambda_0} : \mathcal{P}_n \to \mathcal{P}_n$ by

$$H_{\lambda_0}(p)(\lambda) = p(\lambda - \lambda_0).$$

Lemma 1.1. *Let λ_0 be a given complex number. Then*

$$T_{epi\,(a)}(H_{\lambda_0}(p), \eta + \operatorname{Re}(\lambda_0)) = \{(H_{\lambda_0}(v), \mu) \,:\, (v, \mu) \in T_{epi\,(a)}(p, \eta)\}.$$

Proof. Define the affine transformation $\hat{H}_{\lambda_0} : \mathcal{P}_n \times \mathbb{R} \to \mathcal{P}_n \times \mathbb{R}$ by

$$\hat{H}_{\lambda_0}(p, \mu) = (H_{\lambda_0}(p), \mu + \mathrm{Re}\,(\lambda_0)).$$

Clearly, the mapping $\hat{H}_{\lambda_0}$ is invertible (indeed, $\hat{H}_{\lambda_0}^{-1} = \hat{H}_{-\lambda_0}$). In addition,

$$\hat{H}_{\lambda_0}^{-1}(\mathrm{epi}\,(a)) = \mathrm{epi}\,(a).$$

Therefore, by [RW98, Exercise 6.7] and the invertibility of $\hat{H}_{\lambda_0}$, we have

$$\begin{aligned} T_{\mathrm{epi}\,(a)}(H_{\lambda_0}(p), \mu + \mathrm{Re}\,(\lambda_0)) &= T_{\mathrm{epi}\,(a)}(\hat{H}_{\lambda_0}(p, \mu)) \\ &= \nabla \hat{H}_{\lambda_0}(p, \mu) T_{\mathrm{epi}\,(a)}(p, \mu) \\ &= \{(H_{\lambda_0}(v), \mu) \,:\, (v, \mu) \in T_{\mathrm{epi}\,(a)}(p, \eta)\}. \qquad \square \end{aligned}$$

We now derive a formula for the tangent cone to epi (a) at $(e_{(n,0)}, 0)$. All of our subsequent analysis relies on this key result. The proof is rather long and involved. It is based on an outline provided by Levantovskii [Lev80] for deriving a formula for the tangent cone to the set of stable polynomials.

THEOREM 1.2. *We have* $(v, \eta) \in T_{epi\,(a)}(e_{(n,0)}, 0)$, *with*

$$v = \beta_1 e_{(n-1,0)} + \beta_2 e_{(n-2,0)} + \cdots + \beta_n, \tag{1.6}$$

if and only if

$$Re\,\beta_1 \geq -n\eta, \tag{1.7}$$

$$Re\,\beta_2 \geq 0, \tag{1.8}$$

$$Im\,\beta_2 = 0, \text{ and} \tag{1.9}$$

$$\beta_k = 0 \text{ for } k = 3, \ldots, n. \tag{1.10}$$

Therefore, for $v \in \mathcal{P}_{n-1}$ *given by* (1.6), *we have*

$$da(e_{(n,0)})(v) = \begin{cases} -\frac{Re\,\beta_1}{n} & \text{if (1.8)–(1.10) hold, and} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. We begin by showing that (1.7)–(1.10) and (1.6) imply that (v, η) is an element of the tangent cone $T_{\mathrm{epi}\,(a)}(e_{(n,0)}, 0)$. This is done by constructing a curve in epi (a) converging to $(e_{(n,0)}, 0)$ and having derivative equal to (v, η). Consider the polynomials having coefficients that are polynomials in ξ and given by

$$\begin{aligned} p(\lambda, \xi) &= \left(\lambda + \frac{\beta_1}{n}\xi\right)^{n-2} \left(\lambda + \sqrt{-1}(\beta_2\xi)^{\frac{1}{2}} + \frac{\beta_1}{n}\xi\right) \left(\lambda - \sqrt{-1}(\beta_2\xi)^{\frac{1}{2}} + \frac{\beta_1}{n}\xi\right) \\ &= \left(\lambda^{n-2} + (n-2)\frac{\beta_1}{n}\xi\lambda^{n-3} + o(\xi)\right) \left(\lambda^2 + 2\frac{\beta_1}{n}\xi\lambda + \beta_2\xi + o(\xi)\right) \\ &= \lambda^n + \beta_1\xi\lambda^{n-1} + \beta_2\xi\lambda^{n-2} + o(\xi) \\ &= \lambda^n + \xi v(\lambda) + o(\xi)\ . \end{aligned}$$

Let ξ be real and positive. Then $a(p(\lambda, \xi)) = -\frac{\mathrm{Re}\,(\beta_1)}{n}\xi$. Therefore,

$$\lim_{\xi \searrow 0} \frac{a(p(\lambda, \xi)) - a(\lambda^n)}{\xi} = -\frac{\mathrm{Re}\,(\beta_1)}{n} \leq \eta,$$

which yields the result.

We now show that any element (v,η) in the tangent cone $T_{\mathrm{epi}\,(a)}(e_{(n,0)},0)$ must satisfy (1.7)–(1.10) if v is given the representation (1.6). To this end, we make use of the following norm on $\mathcal{P}_n \times \mathbb{R}$:

$$\left\|(b_0 e_{(n,0)} + b_1 e_{(n-1,0)} + \cdots + b_n, \mu)\right\| = \max\{\, |b_0|\,,\, |b_1|\,, \ldots,\, |b_n|\,,\, |\mu|\,\}\ .$$

Let $(v,\eta) \in T_{\mathrm{epi}\,(a)}(e_{(n,0)},0)$ with v written as in (1.6). By definition there is a sequence $\{(p_k,\mu_k)\} \in \mathrm{epi}\,(a)$ with $(p_k,\mu_k) \to (e_{(n,0)},0)$ and

$$\frac{((p_k,\mu_k) - (e_{(n,0)},0))}{\left\|(p_k,\mu_k) - (e_{(n,0)},0)\right\|} \to (\gamma v, \gamma\eta) \tag{1.11}$$

for some $\gamma > 0$.

Given $\epsilon \in \mathbb{C}^n$, define $\sigma_j : \mathbb{C}^n \to \mathbb{C}$ for $j = 1,2,\ldots,n$ to be the symmetric functions

$$\sigma_1(\epsilon) = \sum_{t=1}^{n} \epsilon_t \quad \text{and} \quad \sigma_j(\epsilon) = \sum_{1\le t_1 < t_2 < \cdots < t_j \le n} \left(\prod_{s=1}^{j} \epsilon_{t_s}\right) \text{ for } j = 2,\ldots,n, \tag{1.12}$$

and set $\sigma = (\sigma_1,\sigma_2,\cdots,\sigma_n)^T$. For each $k = 1,2,\ldots$ there exist complex numbers $\epsilon^k = (\epsilon_{k1},\ \epsilon_{k2},\ldots,\epsilon_{kn})^T \to 0$ such that $\mathrm{Re}\,(\epsilon_{kj}) \ge -\mu_k$ for $j = 1,2,\ldots,n$ and

$$p_k(\lambda) = \prod_{j=1}^{n} (\lambda + \epsilon_{kj}) = (\lambda^n + \sigma_1(\epsilon^k)\lambda^{n-1} + \cdots + \sigma_n(\epsilon^k))\ . \tag{1.13}$$

For each $k = 1,2,\ldots$, set

$$\nu_k = \left\|(p_k,\mu_k) - (e_{(n,0)},0)\right\| = \max\{(\left\|\sigma(\epsilon^k)\right\|_\infty,\, |\mu_k|\,\}.$$

Then the limit (1.11) can be interpreted componentwise as

$$\gamma\beta_j = \lim_{k\to\infty} \frac{\sigma_j(\epsilon^k)}{\nu_k}$$

for $j = 1,2,\ldots,n$. Set $\tilde\sigma_j = \gamma\beta_j$ for $j = 1,2,\ldots,n$. We establish the result by showing that

$$\mathrm{Re}\,\tilde\sigma_1 \ge -n\gamma\eta,\ \ \mathrm{Re}\,\tilde\sigma_2 \ge 0,\ \ \mathrm{Im}\,\tilde\sigma_2 = 0,\ \text{ and } \tilde\sigma_k = 0 \text{ for } k = 3,4,\ldots,n. \tag{1.14}$$

Clearly, $\mathrm{Re}\,(\tilde\sigma_1) \ge -n\gamma\eta$ since $\mathrm{Re}\,(\sigma_1(\epsilon^k)) = \sum_{j=1}^{n} \mathrm{Re}\,(\epsilon_{kj}) \ge -n\mu_k$ for all $k = 1,2,\ldots$ and $\mu_k/\nu_k \to \gamma\eta$. We now show that $\tilde\sigma_j = 0$ for $j = 3,4,\ldots,n$. First note that

$$\sigma_j(\epsilon) = o(\|\epsilon\|_\infty^2) \quad \text{for } j = 3,4,\ldots,n. \tag{1.15}$$

Define

$$\alpha_{kj} = \mathrm{Re}\,\epsilon_{kj} \quad \text{and} \quad \delta_{kj} = \mathrm{Im}\,\epsilon_{kj} \tag{1.16}$$

for $j = 1,2,\ldots,n$ and $k = 1,2,\ldots$. Note that $\alpha_{kj} \ge -\mu_k$ for $j = 1,2,\ldots,n$ and $k = 1,2,\ldots$. In addition, it is easily verified that

$$\mathrm{Re}\,\sigma_2(\epsilon^k) = \sum_{s<t} [\alpha_{ks}\alpha_{kt} - \delta_{ks}\delta_{kt}] \text{ and } \mathrm{Im}\,\sigma_2(\epsilon^k) = \left[\sum_{s<t} \alpha_{ks}\delta_{kt} + \sum_{s<t} \delta_{ks}\alpha_{kt}\right].$$

Then, by definition,

$$\begin{aligned}
\left|\sigma_1(\epsilon^k)\right|^2 &= \left\|\epsilon^k\right\|_2^2 + 2\sum_{s<t} \operatorname{Re}(\bar{\epsilon}_{ks}\epsilon_{kt}) \\
&= \left\|\epsilon^k\right\|_2^2 + 2\sum_{s<t} \alpha_{ks}\alpha_{kt} + 2\sum_{s<t} \delta_{ks}\delta_{kt} \\
&= \left\|\epsilon^k\right\|_2^2 + 4\sum_{s<t} \alpha_{ks}\alpha_{kt} + 2\sum_{s<t} [\delta_{ks}\delta_{kt} - \alpha_{ks}\alpha_{kt}] \\
&= \left\|\epsilon^k\right\|_2^2 + 4\sum_{s<t} \alpha_{ks}\alpha_{kt} - 2\operatorname{Re}\sigma_2(\epsilon^k) \\
&\geq \left\|\epsilon^k\right\|_\infty^2 - 2n(n-1)\mu_k^2 - 2\operatorname{Re}(\sigma_2(\epsilon^k)) \\
&\geq \left\|\epsilon^k\right\|_\infty^2 - 4n(n-1)\max\{\,|\mu_k|\,,\,\left|\sigma_2(\epsilon^k)\right|\,\}\ ,
\end{aligned}$$

whenever $|\mu_k| \leq 1$. Hence, if ϵ^k and μ_k are such that $\left|\sigma_1(\epsilon^k)\right| < \frac{\|\epsilon^k\|_\infty}{2}$ and $|\mu_k| \leq 1$, then, for $\Delta = \frac{3}{16n^2}$, we have

$$\max\{\,|\mu_k|\,,\,\left|\sigma_2(\epsilon^k)\right|\,\} > \Delta\left\|\epsilon^k\right\|_\infty^2 .$$

On the other hand, if $\left|\sigma_1(\epsilon^k)\right| \geq \frac{\|\epsilon^k\|_\infty}{2}$ and $\left\|\epsilon^k\right\|_\infty \leq 1$, then $\left|\sigma_1(\epsilon^k)\right| \geq \frac{\|\epsilon^k\|_\infty^2}{4}$. Thus, in either case, we have

$$\max(\,|\mu_k|\,,\,\left|\sigma_1(\epsilon^k)\right|\,,\,\left|\sigma_2(\epsilon^k)\right|\,) \geq \Delta\left\|\epsilon^k\right\|_\infty^2 , \tag{1.17}$$

whenever $\left\|\epsilon^k\right\|_\infty \leq 1$ and $|\mu_k| \leq 1$. This implies that

$$\nu_k \geq \Delta\left\|\epsilon^k\right\|_\infty^2 \tag{1.18}$$

for all k sufficiently large. This bound, in conjunction with (1.15), allows us to conclude that

$$\tilde{\sigma}_j = \lim_{k\to\infty} \frac{\sigma_j(\epsilon^k)}{\nu_k} = 0 \qquad \text{for } j = 3, 4, \ldots, n.$$

We now turn our attention to the coefficient $\tilde{\sigma}_2$. If

$$\max\{\,\left|\sigma_1(\epsilon^k)\right|\,,\,\left|\sigma_2(\epsilon^k)\right|\,\} = o(\nu_k),$$

we are done since then $\tilde{\sigma} = 0$. Hence, we assume that

$$\max\{\,\left|\sigma_1(\epsilon^k)\right|\,,\,\left|\sigma_2(\epsilon^k)\right|\,\} \neq o(\nu_k)$$

so that

$$\nu_k = \max\{\,\left|\sigma_1(\epsilon^k)\right|\,,\,\left|\sigma_2(\epsilon^k))\right|\,,\,|\mu_k|\,\}$$

for all k sufficiently large. Set $\tilde{\nu}_{kj} = \max\{\,\left|\sigma_j(\epsilon^k)\right|\,,\,|\mu_k|\,\}$ for $j = 1, 2$. Observe that if $\lim_{k\to\infty} \frac{\sigma_2(\epsilon^k)}{\tilde{\nu}_{k1}} = 0$, then we are done since in this case $\nu_k = \tilde{\nu}_{k1}$ for all k sufficiently large which implies that $\tilde{\sigma}_2 = 0$. Hence, with no loss in generality, we can assume that there is a constant $c > 0$ such that

$$\left|\sigma_2(\epsilon^k)\right| \geq c\tilde{\nu}_{k1} \quad \text{for all } k = 1, 2, \ldots. \tag{1.19}$$

Therefore, there is a constant $K > 0$ such that

$$K\left|\sigma_2(\epsilon^k)\right| \geq \nu_k \quad \text{for all } k \text{ sufficiently large.} \tag{1.20}$$

Now observe that

$$|\sigma_2(\epsilon)| = \left|\sum_{s<t} \epsilon_s \epsilon_t\right| \leq \sum_{s<t} |\epsilon_s|\,|\epsilon_t| \leq \frac{n(n-1)}{2}\|\epsilon\|_\infty^2. \tag{1.21}$$

Therefore, for all k sufficiently large,

$$c\left|\operatorname{Re}(\sigma_1(\epsilon^k))\right| \leq c\left|\sigma_1(\epsilon^k)\right| \leq c\tilde{\nu}_{k1} \leq \left|\sigma_2(\epsilon^k)\right| \leq \frac{n(n-1)}{2}\left\|\epsilon^k\right\|_\infty^2,$$

and so, from (1.20), we have

$$|\mu_k| \leq \nu_k \leq \frac{Kn(n-1)}{2}\left\|\epsilon^k\right\|_\infty^2. \tag{1.22}$$

In particular, this implies that

$$\frac{\mu_k}{\|\epsilon^k\|_\infty} \to 0.$$

In addition, since $\alpha_{kj} + \mu_k \geq 0$ for each $j = 1, 2, \ldots, n$ and all $k = 1, 2, \ldots$ and

$$0 \leq \sum_{j=1}^n \frac{\alpha_{kj} + \mu_k}{\|\epsilon^k\|_\infty} \leq \frac{\left|\operatorname{Re}(\sigma_1(\epsilon^k))\right| + n\,|\mu_k|}{\|\epsilon^k\|_\infty} \leq \frac{n(n-1)}{2}\left(\frac{1}{c} + Kn\right)\left\|\epsilon^k\right\|_\infty,$$

for all $k = 1, 2, \ldots$ (recall the definition of the α_{kj}'s from (1.16)), we obtain

$$\lim_{k\to\infty} \frac{\alpha_{kj}}{\|\epsilon^k\|_\infty} = 0 \quad \text{for } j = 1, 2, \ldots, n. \tag{1.23}$$

Putting together the bounds (1.18), (1.20), and (1.22), we obtain the relation

$$\Delta\left\|\epsilon^k\right\|_\infty^2 \leq \nu_k \leq K\left|\sigma_2(\epsilon^k)\right| \leq K\frac{n(n-1)}{2}\left\|\epsilon^k\right\|_\infty^2, \tag{1.24}$$

for all $k = 1, 2, \ldots$. In addition, the bound (1.19) implies that

$$\frac{\left|\operatorname{Im}(\sigma_1(\epsilon^k))\right|^2}{|\sigma_2(\epsilon^k)|} \leq \frac{\left|\sigma_1(\epsilon^k)\right|^2}{|\sigma_2(\epsilon^k)|} \leq \frac{1}{c^2}\left|\sigma_2(\epsilon^k)\right|$$

so that

$$\frac{\left|\operatorname{Im}(\sigma_1(\epsilon^k))\right|^2}{|\sigma_2(\epsilon^k)|} \to 0.$$

Now since $\left|\operatorname{Im}(\sigma_1(\epsilon^k))\right|^2 = \sum_{j=1}^n \delta_{kj}^2 + 2\sum_{s<t}\delta_{ks}\delta_{kt}$, this implies that

$$\lim_{k\to\infty} \sum_{s<t} \frac{\delta_{ks}\delta_{kt}}{|\sigma_2(\epsilon^k)|} \leq 0. \tag{1.25}$$

Finally, recall that

$$\sigma_2(\epsilon^k) = \left[\sum_{s<t} \alpha_{ks}\alpha_{kt} - \sum_{s<t} \delta_{ks}\delta_{kt}\right] + i\left[\sum_{s<t} \alpha_{ks}\delta_{kt} + \sum_{s<t} \delta_{ks}\alpha_{kt}\right].$$

Therefore, by (1.24), (1.25), and (1.23), we see that

$$\mathrm{Re}\,(\tilde{\sigma}_2) = \lim_{k\to\infty} \frac{\mathrm{Re}\,(\sigma_2(\epsilon^k))}{\nu_k} \geq 0.$$

Similarly, from (1.24) and (1.23), we have

$$\begin{aligned} |\,\mathrm{Im}\,(\tilde{\sigma}_2)| &= \lim_{k\to\infty} \frac{\left|\,\mathrm{Im}\,(\sigma_2(\epsilon^k))\right|}{\nu_k} \\ &\leq \Delta^{-1} \lim_{k\to\infty} \sum_{s<t} \left(\frac{|\alpha_{ks}|}{\|\epsilon^k\|_\infty} \frac{|\delta_{kt}|}{\|\epsilon^k\|_\infty} + \frac{|\alpha_{kt}|}{\|\epsilon^k\|_\infty} \frac{|\delta_{ks}|}{\|\epsilon^k\|_\infty} \right) \\ &= 0, \end{aligned}$$

since $\frac{|\delta_{kj}|}{\|\epsilon^k\|_\infty} \leq 1$ for all $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$.

The final statement of the theorem concerning the formula for $da(e_{(n,0)})(v)$ now follows immediately from the equivalence of (1.1) and (1.7)–(1.10). ∎

By combining Lemma 1.1 with Theorem 1.2, we obtain the following corollary.

COROLLARY 1.3. *Given $\lambda_0 \in \mathbb{C}$, we have $(v, \eta) \in T_{epi\,(a)}(e_{(n,\lambda_0)}, Re\,(\lambda_0))$, with*

$$v = \beta_1 e_{(n-1,\lambda_0)} + \beta_2 e_{(n-2,\lambda_0)} + \cdots + \beta_n \ , \tag{1.26}$$

if and only if $\beta_1, \beta_2, \ldots, \beta_n$ satisfy the conditions (1.7)–(1.10). *Therefore, for $v \in \mathcal{P}_{n-1}$ given by* (1.26), *we have*

$$da(e_{(n,\lambda_0)})(v) = \begin{cases} -\frac{Re\,\beta_1}{n} & \textit{if } (1.8)\text{–}(1.10) \textit{ hold,} \\ +\infty & \textit{otherwise.} \end{cases}$$

We now show that the factorization of a polynomial into powers of linear factors (or the *prime factorization*) can be used to obtain a description of the tangent cone to the epigraph of a from Corollary 1.3. We begin by developing a tool that allows us to treat each of the linear factors in the prime factorization separately. We then glue the results for each of the factors back together to obtain a result for the polynomial as a whole. This tool is provided in the next lemma which establishes a local property for factorizations into relatively prime factors.

LEMMA 1.4. *Let $(n_1, n_2, \ldots, n_m)$ be a partition of n, that is, for $j = 1, 2, \ldots, m$ each n_j is a positive integer and $n = \sum_{j=1}^m n_j$. Set*

$$\mathcal{S} = \mathbb{C} \times \mathcal{P}_{n_1-1} \times \mathcal{P}_{n_2-1} \times \cdots \times \mathcal{P}_{n_m-1}$$

and let $p_j \in \mathcal{M}_{n_j}$ for $j = 1, 2, \ldots, m$. Consider the mapping $F : \mathcal{S} \to \mathcal{P}_n$ given by

$$F(v_0, v_1, v_2, \ldots, v_m) = (1 + v_0) \prod_{j=1}^m (p_j + v_j) \ .$$

If the polynomials $p_1, \ldots, p_m$ are relatively prime(i.e., have no common roots), then there exist open neighborhoods U of $0 \in \mathcal{S}$ and W of $F(0) \in \mathcal{P}_n$ such that F is *a homeomorphism between U and W with $\nabla(F^{-1})$ existing, continuous on W, and satisfying $\nabla(F^{-1})(F(u)) = [\nabla F(u)]^{-1}$ for all $u \in U$. Thus, in particular, we have Ran $(\nabla F(0)) = \mathcal{P}_n$; that is, every polynomial $h \in \mathcal{P}_n$ can be written as*

$$h = \nabla F(0)(w_0, w_1, \dots, w_m) = \sum_{j=0}^{m} r_j w_j \ , \tag{1.27}$$

for some $(w_0, w_1, \dots, w_m) \in \mathcal{S}$, where

$$r_0 = \prod_{j=1}^{m} p_j \quad \text{and} \quad r_s = \prod_{j \neq s} p_j \quad \text{for } s = 1, 2, \dots, m. \tag{1.28}$$

Proof. Since $\dim(\mathcal{S}) = n + 1 = \dim(\mathcal{P}_n)$, the result follows from the classical inverse function theorem once it is shown that $\ker(\nabla F(0)) = \{0\}$. Let $\mathcal{Z}_j$ denote the set of zeros of the polynomial p_j counting multiplicity, for $j = 1, 2, \dots, m$, and let $(w_0, w_1, \dots, w_m) \in \ker(\nabla F(0))$. Since the polynomials $p_1, p_2, \dots, p_m$ are relatively prime, we have $\mathcal{Z}_j \cap \mathcal{Z}_s = \emptyset$ whenever $j \neq s$. Equations (1.27) and (1.28) and the inclusion $(w_0, w_1, \dots, w_m) \in \ker(\nabla F(0))$ imply that for each $s = 1, 2, \dots, m$ the polynomial

$$f_s = r_s w_s$$

has zeros not only at the points $\cup_{j \neq s} \mathcal{Z}_j$ (with the corresponding multiplicities) but also at the points $\mathcal{Z}_s$ (with the corresponding multiplicities). Hence, each f_s is either the zero polynomial or its degree is at least n. However, the degree of each f_s is at most $n-1$, since $w_s \in \mathcal{P}_{n_s-1}$. Therefore, f_s is the zero polynomial for $s = 1, 2, \dots, m$. This in turn implies that $w_j = 0$ for $j = 1, 2, \dots, m$, and finally that $w_0 = 0$. Consequently, $\ker(\nabla F(0)) = \{0\}$. $\square$

As a first application of Lemma 1.4, we show if a polynomial is written as a product of relatively prime factors, then the tangent cone to the epigraph of a at this polynomial is contained within a kind of product of the tangent cones associated with each of the relatively prime factors.

THEOREM 1.5. *Let $(n_1, n_2, \dots, n_m)$ be a partition of n, and let $p_j \in \mathcal{M}_{n_j}$ for $j = 1, 2, \dots, m$ be relatively prime. Set $p = \prod_{j=1}^{m} p_j \in \mathcal{M}_n$. Let the space $\mathcal{S}$ and the function $F : \mathcal{S} \to \mathcal{P}_n$ be as given in Lemma 1.4. If $(h, \omega) \in T_{epi\,(a)}(p, a(p))$, then there exists $(0, w_1, w_2, \dots, w_m) \in \mathcal{S}$ such that h is given by (1.27) and (1.28), where, for $j = 1, \dots, m$, $(w_j, \omega) \in T_{epi\,(a_{n_j})}(p_j, a(p))$ and a_{n_j} denotes the abscissa mapping on $\mathcal{P}_{n_j}$.*

Proof. Let (h, ω) be a nonzero element of the tangent cone $T_{\text{epi}\,(a)}(p, a(p))$. Then there is a sequence $\{(q_k, \alpha_k)\} \subset \text{epi}\,(a) \subset \mathcal{M}_n \times \mathbb{R}$ and a scalar $\gamma > 0$ such that

$$(q_k, \alpha_k) \to (p, a(p)) \quad \text{and} \quad \frac{(q_k, \alpha_k) - (p, a(p))}{\|(q_k, \alpha_k) - (p, a(p))\|} \to (\gamma h, \gamma \omega) \ .$$

Let $F : \mathcal{S} \to \mathcal{P}_n$ be as in Lemma 1.4. Then, by trimming finitely many terms from the beginning of the sequence if necessary so that q_k is sufficiently close to p, Lemma 1.4 yields the existence of a sequence $\{(0, v_{k1}, v_{k2}, \dots, v_{km})\} \subset \mathcal{S}$ such that $(0, v_{k1}, v_{k2}, \dots, v_{km}) \to 0$ and

$$q_k = F(0, v_{k1}, v_{k2}, \dots, v_{km}) = \prod_{j=1}^{m} (p_j + v_{kj}) \quad \text{for all } k = 1, 2, \dots ,$$

since $\{q_k\} \subset \mathcal{M}_n = \operatorname{dom}(a)$. Since $(q_k, \alpha_k) \in \operatorname{epi}(a)$, we have

$$(p_j + v_{kj}, \alpha_k) \in \operatorname{epi}(a_{n_j}) \quad \text{for all } j = 1, 2, \ldots, m \text{ and } k = 1, 2, \ldots \tag{1.29}$$

and

$$(p_j + v_{kj}, \alpha_k) \to (p_j, a(p)) \quad \text{for all } j = 1, 2, \ldots, m. \tag{1.30}$$

Set $v^k = (0, v_{k1}, v_{k2}, \ldots, v_{km})$ for $k = 1, 2, \ldots$, and set $\bar{v} = 0$ so that $v^k \to \bar{v}$. Then

$$\begin{aligned} q_k - p &= F(v^k) - F(\bar{v}) \\ &= \nabla F(\bar{v})(v^k - \bar{v}) + o(\|v^k - \bar{v}\|) \ . \end{aligned} \tag{1.31}$$

By Lemma 1.4, $\nabla(F^{-1})$ is continuous in a neighborhood of p so that F^{-1} is Lipschitz continuous near p. Consequently, there is a constant $K > 0$ such that $\|v^k - \bar{v}\| \leq K\|q_k - p\|$ for all $k = 1, 2, \ldots$. This fact, combined with (1.31), yields

$$\begin{aligned} \gamma h &= \lim_{k\to\infty} \frac{q_k - p}{\|(q_k, \alpha_k) - (p, a(p))\|} \\ &= \nabla F(\bar{v}) \lim_{k\to\infty} \frac{v^k - \bar{v}}{\|(q_k, \alpha_k) - (p, a(p))\|} \\ &= \nabla F(0)(0, \hat{w}_1, \hat{w}_2, \ldots, \hat{w}_m), \end{aligned} \tag{1.32}$$

where

$$\hat{w}_j = \lim_{k\to\infty} \frac{v_{kj}}{\|(q_k, \alpha_k) - (p, a(p))\|} \quad \text{for } j = 1, 2, \ldots, m.$$

Equation (1.32) verifies (1.27) with $w_j = \gamma^{-1}\hat{w}_j$ for $j = 1, 2, \ldots, m$. From (1.29), (1.30), and definition (1.2) (here $t_k = \|(q_k, \alpha_k) - (p, a(p))\|$), we have that (w_j, ω) is an element of $T_{\operatorname{epi}(a_{n_j})}(p_j, a(p))$, for $j = 1, 2, \ldots, m$, which proves the result. ∎

We now apply Corollary 1.3, Lemma 1.4, and Theorem 1.5 to obtain a complete representation of the tangent cone to the epigraph of the abscissa mapping at an arbitrary polynomial. This representation involves the prime factorization of the polynomial. For this purpose, and for the application of this result in later sections, it is useful to introduce some more notation.

Let $p \in \mathcal{M}_n$ have prime factorization

$$p = \prod_{j=1}^{m} e_{(n_j, \lambda_j)}, \tag{1.33}$$

where $\lambda_1, \ldots, \lambda_m$ are distinct complex numbers and $(n_1, n_2, \ldots, n_m)$ is a partition of n. Define $\mathcal{S}_p$ to be the product space

$$\mathcal{S}_p = \mathbb{C} \times \mathcal{P}_{n_1 - 1} \times \mathcal{P}_{n_2 - 1} \times \cdots \times \mathcal{P}_{n_m - 1}. \tag{1.34}$$

In conjunction with $\mathcal{S}_p$, we define the mapping $F_p : \mathcal{S}_p \to \mathcal{P}_n$ by

$$F_p(v_0, v_1, \ldots, v_m) = (1 + v_0) \prod_{j=1}^{m} (e_{(n_j, \lambda_j)} + v_j) \quad \text{for all } (v_0, v_1, \ldots, v_m) \in \mathcal{S}_p, \tag{1.35}$$

so that $F_p(0) = p$. By analogy with (1.27), for every $(w_0, w_1, \ldots, w_m) \in \mathcal{S}$, we have

$$\nabla F_p(0)(w_0, w_1, \ldots, w_m) = \sum_{j=0}^{m} r_j w_j \ , \tag{1.36}$$

where

$$r_0 = p \quad \text{and} \quad r_s = \prod_{j \neq s} e_{(n_j, \lambda_j)} \quad \text{for } s = 1, 2, \ldots, m. \tag{1.37}$$

In addition, we define

$$\mathcal{I}(p) = \{ j \in \{1, 2, \ldots, m\} \mid a(p) = \operatorname{Re} \lambda_j \} , \tag{1.38}$$

the set of indices of active roots of p.

We now state and prove the main result of this section.

THEOREM 1.6. *Let $p \in \mathcal{M}_n$ have factorization* (1.33). *Then (h, ω) is an element of $T_{epi\,(a)}(p, a(p))$ if and only if there exists a vector $(w_0, w_1, w_2, \ldots, w_m) \in \mathcal{S}_p$ such that*

$$h = \nabla F_p(0)(w_0, w_1, w_2, \ldots, w_m), \tag{1.39}$$

where $\nabla F_p(0)$ is defined in (1.36)–(1.37),

$$w_0 = 0, \tag{1.40}$$

and

$$(w_j, \omega) \in T_{epi\,(a_{n_j})}(e_{(n_j, \lambda_j)}, a(p)) \quad \textit{for } j = 1, 2, \ldots, m. \tag{1.41}$$

In addition, if for each $j = 1, 2, \ldots, m$, w_j is given the representation

$$w_j = \beta_{j1} e_{(n_j - 1, \lambda_j)} + \beta_{j2} e_{(n_j - 2, \lambda_j)} + \cdots + \beta_{jn_j} \ , \tag{1.42}$$

then, for each $j \in \mathcal{I}(p)$, a necessary and sufficient condition for (1.41) *to hold is that*

$$Re\, \beta_{j1} \geq -n_j \omega, \tag{1.43}$$

$$Re\, \beta_{j2} \geq 0, \tag{1.44}$$

$$Im\, \beta_{j2} = 0, \ \textit{and} \tag{1.45}$$

$$\beta_{js} = 0 \ \textit{for } s = 3, 4, \ldots, n_j \ . \tag{1.46}$$

Proof. Let us first assume that $(h, \omega) \in T_{\text{epi}\,(a)}(p, a(p))$ and show that (h, ω) must satisfy (1.39), (1.40), and (1.41). By Lemma 1.4, there must exist a vector $(w_0, w_1, w_2, \ldots, w_m)$ in $\mathcal{S}_p$ such that (1.39) holds. The fact that $w_0 = 0$ follows from (1.4), while (1.41) follows immediately from Theorem 1.5. The conditions (1.43)–(1.46) follow from (1.41) and Corollary 1.3.

Next, let us assume that $(h, \omega) \in \mathcal{P}_{n-1} \times \mathbb{R}$ satisfies (1.39), (1.40), and (1.41). We need to show that $(h, \omega) \in T_{\text{epi}\,(a)}(p, a(p))$. We accomplish this by following the approach taken in Theorem 1.2. That is, we will exhibit a curve in epi (a) passing through $(p, a(p))$ and having the tangent direction (h, ω) at $(p, a(p))$. For $j = 1, 2, \ldots, m$, give each w_j in (1.39) the representation (1.42). Then, by Corollary 1.3, we know that the conditions (1.43)–(1.46) are satisfied for $j \in \mathcal{I}(p)$. For each such $j \in \mathcal{I}(p)$, define

$$\begin{aligned} p_j(\lambda,\xi) &= \left((\lambda-\lambda_j)+\frac{\beta_{j1}}{n_j}\xi\right)^{n_j-2} \\ &\quad\times\left((\lambda-\lambda_j)+\sqrt{-1}(\beta_{j2}\xi)^{\frac{1}{2}}+\frac{\beta_{j1}}{n_j}\xi\right)\left((\lambda-\lambda_j)-\sqrt{-1}(\beta_{j2}\xi)^{\frac{1}{2}}+\frac{\beta_{j1}}{n_j}\xi\right) \\ &= (\lambda-\lambda_j)^{n_j}+(\beta_{j1}\xi)(\lambda-\lambda_j)^{n_j-1}+\beta_{j2}\xi(\lambda-\lambda_j)^{n_j-2}+o(\xi) \\ &= (\lambda-\lambda_j)^{n_j}+\xi w_j(\lambda)+o(\xi), \end{aligned} \tag{1.47}$$

and, for $j \in \{1,2,\dots,m\}\backslash\mathcal{I}(p)$, define

$$p_j(\lambda,\xi) = (\lambda-\lambda_j)^{n_j}+\xi w_j(\lambda)\ . \tag{1.48}$$

Set $p(\lambda,\xi) = \prod_{j=1}^m p_j(\lambda,\xi)$, so that, from (1.36), (1.37), and (1.39),

$$\begin{aligned} p(\lambda,\xi) &= p(\lambda)+\xi\sum_{j=1}^m r_j(\lambda)w_j(\lambda)+o(\xi) \\ &= p(\lambda)+\xi\nabla F_p(0)(0,w_1,\dots,w_m)(\lambda)+o(\xi) \\ &= p(\lambda)+\xi h(\lambda)+o(\xi). \end{aligned}$$

Then, for all ξ small, positive, and real,

$$\begin{aligned} a(p(\lambda,\xi)) &= \max_{j\in\mathcal{I}(p)} \operatorname{Re}\left(\lambda_j-\frac{\beta_{j1}}{n_j}\xi\right) \\ &\le a(p)+\xi\omega \end{aligned}$$

so that $(p(\lambda,\xi), a(p)+\xi\omega) \in \operatorname{epi}(a)$ for all ξ small, positive, and real. Therefore, since

$$\lim_{\xi\searrow 0}\frac{(p(\lambda,\xi),a(p)+\xi\omega)-(p(\lambda),a(p))}{\xi} = (h(\lambda),\omega),$$

we have $(h,\omega) \in T_{\operatorname{epi}(a)}(p,a(p))$, completing the proof. ∎

Corollary 1.7. *Let $p \in \mathcal{M}_n$ have factorization* (1.33) *and let $h \in \mathcal{P}_n$. By Lemma* 1.4, *there exists $(w_0,w_1,w_2,\dots,w_m)$ in $\mathcal{S}_p$ such that* (1.39) *holds, where, for each $j=1,2,\dots,m$, w_j can be written as in* (1.42). *With this representation for h, either $w_0=0$ and* (1.44)–(1.46) *hold for $j \in \mathcal{I}(p)$, in which case*

$$da(p)(h) = \max_{j\in\mathcal{I}(p)}\frac{-\,Re\,(\beta_{j1})}{n_j}\ ,$$

or

$$da(p)(h) = +\infty.$$

Proof. By Theorem 1.6, we know that $da(p)(h) = +\infty$ if either $w_0 \neq 0$ or the coefficients $\beta_{js},\ s=1,2,\dots,n_j$, do not satisfy one of the conditions in (1.44)–(1.46) for every $j \in \mathcal{I}(p)$. On the other hand, if $w_0=0$ and all of the conditions in (1.44)–(1.46) are satisfied for every $j \in \mathcal{I}(p)$, then the inequality (1.43) in Theorem 1.6 implies that $(h,\omega) \in T_{\operatorname{epi}(a)}((p,a(p)))$ if and only if $\omega \ge \frac{-\operatorname{Re}(\beta_{j1})}{n_j}$ for every $j \in \mathcal{I}(p)$. Since $T_{\operatorname{epi}(a)}((p,a(p))) = \operatorname{epi}(da(p))$ [RW98, Theorem 8.2], this proves the corollary. ∎

2. Regular subgradients and the normal cone. We now turn our attention to the variational objects *dual* to the subderivative and the tangent cone. These are the subgradients and the normal cone. These objects are defined in terms of a duality pairing between the linear space $\mathcal{P}_n$ and its *dual space*. Traditionally the *dual space* is the space of continuous linear functionals on the *primal* space (which in our setting is $\mathcal{P}_n$). The *duality pairing* is then the continuous bilinear functional obtained by evaluating a given linear functional at a given point. However, in general, the dual space may have many possible representations and for each representation there may be any number of bilinear functionals that *pair the spaces in duality.*

In our analysis, we have chosen to regard $\mathcal{P}_n$ as a Hilbert space, in which case the dual of $\mathcal{P}_n$ is itself. However, we will need to consider a whole family of duality pairings, or inner products, on $\mathcal{P}_n$. To describe this family of inner products, recall that for each $\lambda_0 \in \mathbb{C}$, the polynomials

$$e_{(j,\lambda_0)},\ j = 0, 1, \ldots, n, \tag{2.1}$$

form a basis for $\mathcal{P}_n$. Hence, for each $\lambda_0 \in \mathbb{C}$, we can define a real inner product on $\mathcal{P}_n$ associated with the representation in this basis. Given $p = \sum_{j=1}^n a_j e_{(n-j,\lambda_0)}$ and $q = \sum_{j=1}^n b_j e_{(n-j,\lambda_0)}$, define the inner product

$$\langle \cdot\,, \cdot \rangle_{(n,\lambda_0)} : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{R}$$

by

$$\langle p\,, q \rangle_{(n,\lambda_0)} = \operatorname{Re} \sum_{j=0}^{n} \bar{a}_j b_j. \tag{2.2}$$

Thus, in the case $n = 0$, we recover the real inner product on $\mathbb{C}$. Note that this family of inner products behaves continuously in p, q, and λ_0 in the sense that the mapping

$$(p, q, \lambda) \mapsto \langle p\,, q \rangle_{(n,\lambda)} \tag{2.3}$$

is continuous on $\mathcal{P}_n \times \mathcal{P}_n \times \mathbb{C}$. To see this, note that the expansions of the polynomials p and q in the basis (2.1) are just their Taylor series expansions at λ_0, hence,

$$\langle p\,, q \rangle_{(n,\lambda)} = \operatorname{Re} \left(\sum_{j=0}^{n} \overline{\frac{p^{(j)}(\lambda)}{j!}} \frac{q^{(j)}(\lambda)}{j!} \right),$$

where $f^{(j)}$ denotes the jth derivative of the function f.

By setting $\lambda_0 = 0$ in (2.1), we obtain the *standard basis* for $\mathcal{P}_n$. The inner product (2.2) associated with the standard basis is simply written $\langle \cdot\,, \cdot \rangle$.

The spaces $\mathcal{S}_p$ defined in (1.34) also play a key role in our analysis; therefore, we need an inner product on these spaces as well. We use the inner product

$$\langle (u_0, u_1, \ldots, u_m)\,, (v_0, v_1, \ldots, v_m) \rangle_p = \sum_{s=0}^{m} \langle u_s\,, v_s \rangle_{(n_s-1,\lambda_s)}\ , \tag{2.4}$$

for every $(u_0, u_1, \ldots, u_m)$ and $(v_0, v_1, \ldots, v_m)$ in $\mathcal{S}_p$, where we define $n_0 = 1$ in this expression and hereafter.

Spaces paired in duality give rise to the notion of the adjoint of a linear transformation. Suppose (X, X^*) and (Y, Y^*) are spaces paired in duality, with the duality pairing between X and X^*, and Y and Y^* given by $\langle \cdot , \cdot \rangle_X$ and $\langle \cdot , \cdot \rangle_Y$, respectively. If A is a linear transformation mapping X to Y, then the adjoint of A, denoted A^*, is the linear transformation mapping Y^* to X^* defined by the condition that

$$\langle A^* y , x \rangle_X = \langle y , Ax \rangle_Y \quad \text{for all } x \in X \text{ and } y \in Y^*.$$

The dual variational objects studied in this section are the cone of regular normals and the set of regular subgradients. The cone of regular normal vectors to the epigraph of a at a point $(p, \mu) \in \operatorname{epi}(a)$, denoted $\widehat{N}_{\operatorname{epi}(a)}(p, \mu)$, is given by

$$\left\{ (z, \eta) \;\middle|\; \begin{array}{c} \langle (z, \eta) , (q, \tau) - (p, \mu) \rangle \le o(\|(q, \tau) - (p, \mu)\|) \\ \forall\, (q, \tau) \in \operatorname{epi}(a) \end{array} \right\},$$

where $\langle (z, \eta) , (q, \tau) \rangle = \eta\tau + \langle z , q \rangle$ (note that $\operatorname{epi}(a) \subset \mathcal{P}_n \times \mathbb{R}$ so that η and τ are real). The cone of regular normals is defined to be the empty set at points not in the epigraph of a. The set of regular subgradients of a at $p \in \operatorname{dom} a = \mathcal{M}_n$ is given by

$$\hat{\partial} a(p) = \{ z \mid a(q) \ge a(p) + \langle z , q - p \rangle + o(\|q - p\|) \;\forall\, q \in \mathcal{P}_n \}.$$

If $p \ne \mathcal{M}_n$, we define $\hat{\partial} a(p)$ to be the empty set. By [RW98, Theorem 8.9], we have the following relationship between the cone of regular normals and the set of regular subgradients:

$$\hat{\partial} a(p) = \left\{ z \;\middle|\; (z, -1) \in \widehat{N}_{\operatorname{epi}(a)}(p, a(p)) \right\}. \tag{2.5}$$

In addition, [RW98, Proposition 6.5] tells us that the cone of regular normals is the polar of the tangent cone at points $(p, a(p)) \in \operatorname{epi}(a)$:

$$\widehat{N}_{\operatorname{epi}(a)}(p, a(p)) = T_{\operatorname{epi}(a)}(p, a(p))^\circ , \tag{2.6}$$

where

$$T_{\operatorname{epi}(a)}(p, a(p))^\circ = \left\{ (z, \xi) \;\middle|\; \langle (z, \xi) , (h, \omega) \rangle \le 0 \;\forall\, (h, \omega) \in T_{\operatorname{epi}(a)}(p, a(p)) \right\}.$$

We take a moment to observe two important consequences of the equivalence (2.6). These observations are based on the relations (1.4) and (1.5). By (1.4), we have that the vector $(e_{(n,0)}, 0)$ is orthogonal to every vector in $T_{\operatorname{epi}(a)}(p, a(p))$, regardless of the choice of the polynomial $p \in \mathcal{M}_n$. Therefore, by (2.6),

$$\left\{ (\beta e_{(n,0)}, 0) \mid \beta \in \mathbb{C} \right\} \subset \widehat{N}_{\operatorname{epi}(a)}(p, a(p)) \quad \text{for every } p \in \mathcal{M}_n. \tag{2.7}$$

In addition, (1.5) and (2.6) imply that

$$\left\{ (\beta e_{(n,0)}, 0) \mid \beta \in \mathbb{C} \right\} = \widehat{N}_{\operatorname{epi}(a)}(p, \mu), \quad \text{whenever } \mu > a(p). \tag{2.8}$$

We now proceed to derive an expression for $\widehat{N}_{\operatorname{epi}(a)}(p, a(p))$ using (2.6) and Theorem 1.6. We then use the relation (2.5) to determine $\hat{\partial} a(p)$.

Theorem 2.1. *Let $p \in \mathcal{M}_n$ have factorization* (1.33) *and let $\mathcal{I}(p)$ be as defined in* (1.38). *Then (z, η) is an element of the normal cone $\widehat{N}_{epi\,(a)}(p, a(p))$ if and only if*

$$\eta \le 0 \tag{2.9}$$

and the vector $u \in \mathcal{S}_p$ defined by $u = \nabla F_p(0)^* z$ and given the representation

$$u_j = \sum_{l=1}^{n_j} \mu_{jl} e_{(n_j - l, \lambda_j)} \quad \text{for } j = 1, \ldots, m \tag{2.10}$$

satisfies

$$u_j = 0 \quad \text{for } j \notin \mathcal{I}(p) \text{ and } j \neq 0, \tag{2.11}$$

$$\mathrm{Re}\, \mu_{j1} \leq 0 \text{ and } \mathrm{Im}\, \mu_{j1} = 0 \quad \text{for } j \in \mathcal{I}(p), \tag{2.12}$$

$$\mathrm{Re}\, \mu_{j2} \leq 0 \quad \text{for } j \in \mathcal{I}(p), \quad \text{and} \tag{2.13}$$

$$\sum_{j \in \mathcal{I}(p)} n_j \mu_{j1} = \eta. \tag{2.14}$$

Proof. Let $(h, \omega) \in T_{\mathrm{epi}\,(a)}(p, a(p))$. By Theorem 1.6, we know that there exists $(0, w_1, w_2, \ldots, w_m) \in \mathcal{S}_p$ such that $h = \nabla F_p(0)(0, w_1, w_2, \ldots, w_m)$, where for $j = 1, 2, \ldots, m$ each w_j has the representation (1.42) with the coefficients β_{js} satisfying (1.43)–(1.46) for $j \in \mathcal{I}(p)$, and, for $j \notin \mathcal{I}(p)$,

$$\beta_{js}, \quad s = 1, 2, \ldots, n_j, \text{ are unrestricted.} \tag{2.15}$$

Now let $(z, \eta) \in \mathcal{P}_n \times \mathbb{R}$ and set $u = (u_0, u_1, \ldots, u_m) = \nabla F_p(0)^* z$, where each u_j, $j = 1, \ldots, m$ is given the representation (2.10). Then, from definition (2.4), we have

$$\begin{aligned}
\langle (z, \eta), (h, \omega) \rangle &= \eta\omega + \langle z, h \rangle \\
&= \eta\omega + \langle z, \nabla F_p(0)(0, w_1, w_2, \ldots, w_m) \rangle \\
&= \eta\omega + \langle \nabla F_p(0)^* z, (0, w_1, w_2, \ldots, w_m) \rangle_p \\
&= \eta\omega + \langle (u_0, u_1, \ldots, u_m), (0, w_1, w_2, \ldots, w_m) \rangle_p \\
&= \eta\omega + \sum_{j=1}^{m} \langle u_j, w_j \rangle_{(n_j - 1, \lambda_j)} \\
&= \eta\omega + \sum_{j=1}^{m} \sum_{l=1}^{n_j} \mathrm{Re}\, \bar{\mu}_{jl} \beta_{jl}.
\end{aligned} \tag{2.16}$$

Hence, by (2.6), $(z, \eta) \in \widehat{N}_{\mathrm{epi}\,(a)}(p, a(p))$ if and only if

$$\eta\omega + \sum_{j=1}^{m} \sum_{l=1}^{n_j} \mathrm{Re}\, \bar{\mu}_{jl} \beta_{jl} \leq 0 \tag{2.17}$$

for all choices of ω and β_{jl}, $j = 1, \ldots, m$, $l = 1, \ldots, n_j$, satisfying (1.43)–(1.46) for each $j \in \mathcal{I}(p)$.

We first show that any $(z, \eta) \in \mathcal{P}_n \times \mathbb{R}$ for which the associated vector $u = (u_0, u_1, \ldots, u_m) = \nabla F_p(0)^* z$, where each u_j, $j = 1, \ldots, m$, has representation (2.10) and for which η and μ_{jl}, $j = 1, \ldots, m$, $l = 1, \ldots, n_j$, satisfy (2.9) and (2.11)–(2.14) is necessarily an element of the normal cone $\widehat{N}_{\mathrm{epi}\,(a)}(p, a(p))$. For this purpose, suppose that ω and β_{jl}, $j = 1, \ldots, m$, $l = 1, \ldots, n_j$ satisfy (1.43)–(1.46) for each $j \in \mathcal{I}(p)$ so that the corresponding vector (h, ω) is an element of the tangent cone $T_{\mathrm{epi}\,(a)}(p, a(p))$.

Then

$$\begin{aligned}\langle (z,\eta),(h,\omega)\rangle &= \eta\omega + \sum_{j=1}^{m}\sum_{l=1}^{n_j} \operatorname{Re}\bar{\mu}_{jl}\beta_{jl}\\ &= \eta\omega + \sum_{j\in\mathcal{I}(p)} [\mu_{j1}\operatorname{Re}\beta_{j1} + \operatorname{Re}\mu_{j2}\operatorname{Re}\beta_{j2}]\\ &\le \eta\omega - \sum_{j\in\mathcal{I}(p)} n_j\mu_{j1}\left(\frac{-\operatorname{Re}\beta_{j1}}{n_j}\right)\\ &\le \eta\omega - \sum_{j\in\mathcal{I}(p)} n_j\mu_{j1}\omega\\ &= 0,\end{aligned}$$

where the first equality follows from (2.16), the second equality from (2.11), (2.12), (1.45), and (1.46), the first inequality from (2.13) and (1.44), the second inequality from (1.43), and the final equality from (2.14). Therefore, the set of (z,η) satisfying (2.9)–(2.14) is contained in $\widehat{N}_{\mathrm{epi}\,(a)}(p,a(p))$.

We now show the reverse inclusion. Let $(z,\eta)\in\widehat{N}_{\mathrm{epi}\,(a)}(p,a(p))$ and set $u=(u_0,u_1,\ldots,u_m)=\nabla F_p(0)^*z$ with each u_j, $j=1,\ldots,m$ given representation (2.10). We show that (z,η) must satisfy the conditions (2.11)–(2.14) by requiring that the inequality (2.17) holds for every (h,ω) in the tangent cone $T_{\mathrm{epi}\,(a)}(p,a(p))$. To this end, let (h,ω) be any element of the tangent cone $T_{\mathrm{epi}\,(a)}(p,a(p))$ so that the corresponding vectors w_j, $j=1,\ldots,m$, satisfy (1.43)–(1.46) for each $j\in\mathcal{I}(p)$ and (2.15) for $j\notin\mathcal{I}(p)$. By setting $\omega=1$ and all β_{jl} equal to zero in (2.17), we find that $\eta\le 0$. By (2.15), β_{js} is free for $j\notin\mathcal{I}(p)$, $s=1,2,\ldots,n_j$, so that (2.17) implies that (2.11) holds. Since $\operatorname{Im}\beta_{j1}$ is free whenever $j\in\mathcal{I}(p)$, (2.17) implies that $\operatorname{Im}\mu_{j1}=0$ for all $j\in\mathcal{I}(p)$. In addition, (1.43) and (2.17) imply that $\operatorname{Re}\mu_{j2}\le 0$ for all $j\in\mathcal{I}(p)$. Therefore, (2.9), (2.11), the second half of (2.12) (i.e., the equality), and (2.13) have been verified.

We now establish the first half of (2.12) (i.e., the inequality) and (2.14). By taking $\operatorname{Re}\beta_{j2}=0$ for $j\in\mathcal{I}(p)$, the expression (2.16) can be simplified to

$$\langle (z,\eta),(h,\omega)\rangle = \eta\omega + \sum_{j\in\mathcal{I}(p)} \mu_{j1}\operatorname{Re}\beta_{j1}. \tag{2.18}$$

By combining this with (2.17), we must have

$$\sum_{j\in\mathcal{I}(p)} \mu_{j1}\operatorname{Re}\beta_{j1} \le -\eta\omega \tag{2.19}$$

for all choices of ω and $\operatorname{Re}\beta_{j1}$, $j\in\mathcal{I}(p)$, satisfying (1.43). Observe that (1.43) holds if and only if

$$\omega \ge \max_{j\in\mathcal{I}(p)} \frac{-\operatorname{Re}\beta_{j1}}{n_j}. \tag{2.20}$$

Since $-\eta\ge 0$, we can multiply this inequality through by $-\eta$ to obtain the inequality

$$-\eta\omega \ge -\eta \max_{j\in\mathcal{I}(p)} \frac{-\operatorname{Re}\beta_{j1}}{n_j}. \tag{2.21}$$

Since the right-hand side of this inequality yields the smallest possible value of the product $-\eta\omega$, we find that (1.43) and (2.19) hold if and only if

$$\sum_{j\in\mathcal{I}(p)} \mu_{j1} \operatorname{Re}\beta_{j1} \le (-\eta) \max_{j\in\mathcal{I}(p)} \frac{-\operatorname{Re}\beta_{j1}}{n_j} \qquad \forall \beta_{j1} \in \mathbb{C},\ j \in \mathcal{I}(p). \tag{2.22}$$

Consider two cases: $\eta = 0$ and $\eta < 0$. If $\eta = 0$, then (2.22) implies that $\mu_{j1} = 0$ for all $j \in \mathcal{I}(p)$ so that (2.12) and (2.14) are satisfied. On the other hand, if $\eta < 0$, define $\tilde{\mu}_j = \frac{n_j}{\eta}\mu_{j1}$ and $\tilde{\beta}_j = \frac{-\operatorname{Re}\beta_{j1}}{n_j}$ for $j \in \mathcal{I}(p)$. Substituting into (2.22), we obtain

$$\sum_{j\in\mathcal{I}(p)} \tilde{\mu}_j \tilde{\beta}_j \le \max_{j\in\mathcal{I}(p)} \tilde{\beta}_j \qquad \forall \tilde{\beta}_j \in \mathbb{R}. \tag{2.23}$$

But this holds if and only if $\tilde{\mu}_j \ge 0$ for $j \in \mathcal{I}(p)$ and $\sum_{j\in\mathcal{I}(p)} \tilde{\mu}_j = 1$, or equivalently, (2.12) and (2.14) hold. ◻

Theorem 2.1 and (2.5) immediately yield the following representation for the set of regular subgradients.

THEOREM 2.2. *Let $p \in \mathcal{M}_n$ have factorization* (1.33). *Then $z \in \hat{\partial} a(p)$ if and only if the vector of polynomials*

$$\nabla F_p(0)^* z = (u_0, u_1, \ldots, u_m) \in \mathcal{S}_p,$$

with

$$u_j = \sum_{l=1}^{n_j} \mu_{jl} e_{(n_j - l, \lambda_j)}, \qquad j = 1, 2, \ldots, m,$$

is such that

$$\begin{aligned} u_j = 0 &\quad \textit{for } j \notin \mathcal{I}(p) \textit{ and } j \ne 0,\\ \textit{Re}\,\mu_{j1} \le 0 \textit{ and } \textit{Im}\,\mu_{j1} = 0 &\quad \textit{for } j \in \mathcal{I}(p),\\ \textit{Re}\,\mu_{j2} \le 0 &\quad \textit{for } j \in \mathcal{I}(p), \textit{ and}\\ \sum_{j\in\mathcal{I}(p)} n_j \mu_{j1} = -1. & \end{aligned}$$

A more concise representation for the set of regular subgradients is possible. First note that if $p = e_{(n,\lambda_0)}$, then, for $(h_0, h_1) \in \mathcal{S}_p = \mathbb{C} \times \mathcal{P}_{n-1}$,

$$\nabla F_p(0)(h_0, h_1) = h_0 e_{(n,\lambda_0)} + h_1$$

and

$$\nabla F_p(0)^* \sum_{j=0}^{n} b_j e_{(n-j,\lambda_0)} = \left(b_0, \sum_{j=1}^{n} b_j e_{(n-j,\lambda_0)} \right), \tag{2.24}$$

since

$$\left\langle \sum_{j=0}^{n} b_j e_{(n-j,\lambda_0)}, h_0 e_{(n,\lambda_0)} + h_1 \right\rangle_{(n,\lambda_0)} = \left\langle \left(b_0, \sum_{j=1}^{n} b_j e_{(n-j,\lambda_0)}\right), (h_0, h_1) \right\rangle_p .$$

In this case $\nabla F_p(0)^* = \nabla F_p(0)^{-1}$. Hence, by Theorem 2.2, we have the following formula for the set of regular subgradients of a at $e_{(n,\lambda_0)}$:

$$\hat{\partial} a(e_{(n,\lambda_0)}) = \left\{ z \;\middle|\; \begin{array}{c} z = \sum_{j=0}^{n} \mu_j e_{(n-j,\lambda_0)}, \\ \text{where } \mu_j \in \mathbb{C},\ j = 0,1,\dots,n, \\ \mu_1 = \frac{-1}{n},\ \text{and } \operatorname{Re}(\mu_2) \le 0 \end{array} \right\}. \tag{2.25}$$

In the general case a similar formula can be obtained with the aid of the recession cone of the set $\hat{\partial} a(e_{(n,\lambda_0)})$:

$$\hat{\partial} a(e_{(n,\lambda_0)})^{\infty} = \left\{ z \;\middle|\; \begin{array}{c} z = \sum_{j=0}^{n} \mu_j e_{(n-j,\lambda_0)}, \\ \text{where } \mu_j \in \mathbb{C},\ j = 0,1,\dots,n, \\ \mu_1 = 0,\ \text{and } \operatorname{Re}(\mu_2) \le 0 \end{array} \right\}. \tag{2.26}$$

Define $\hat{\partial} a(e_{(n,\lambda_0)})^{\tilde{\infty}}$ as the projection of $\hat{\partial} a(e_{(n,\lambda_0)})^{\infty}$ onto $\mathcal{P}_{n-1}$:

$$\hat{\partial} a(e_{(n,\lambda_0)})^{\tilde{\infty}} = \left\{ z \;\middle|\; \begin{array}{c} z = \sum_{j=1}^{n} \mu_j e_{(n-j,\lambda_0)}, \\ \text{where } \mu_j \in \mathbb{C},\ j = 1,\dots,n, \\ \mu_1 = 0,\ \text{and } \operatorname{Re}(\mu_2) \le 0 \end{array} \right\}.$$

Then, given a polynomial $p \in \mathcal{M}_n$ having prime factorization (1.33), the set of regular subgradients of a at p has the form

$$\hat{\partial} a(p) = \nabla F_p(0)^{-*} \left[\operatorname{conv} \{ v_j \mid j \in \mathcal{I}(p) \} + K \right], \tag{2.27}$$

where $v_1, \dots, v_m \in \mathcal{S}_p$ are given by

$$\begin{aligned} v_1 &= -\frac{1}{n_1}(0, e_{(n_1-1,\lambda_1)}, 0, \dots, 0), \\ v_2 &= -\frac{1}{n_2}(0, 0, e_{(n_2-1,\lambda_2)}, 0, \dots, 0), \\ &\vdots \\ v_m &= -\frac{1}{n_m}(0, 0, \dots, 0, e_{(n_m-1,\lambda_m)}), \end{aligned}$$

and K is the convex cone in $\mathcal{S}_p$ given by

$$K = \mathbb{C} \times \hat{\partial} a(e_{(n_1,\lambda_1)})^{\tilde{\infty}} \times \cdots \times \hat{\partial} a(e_{(n_m-1,\lambda_m)})^{\tilde{\infty}}.$$

Observe that this implies the recession cone of $\hat{\partial} a(p)$ is given by

$$\hat{\partial} a(p)^{\infty} = \nabla F_p(0)^{-*} K. \tag{2.28}$$

3. Subdifferential regularity. The set of normal vectors to $\operatorname{epi}(a)$ at a point $(p,\mu) \in \operatorname{epi}(a)$ is given by

$$N_{\operatorname{epi}(a)}(p,\mu) = \left\{ (z,\omega) \;\middle|\; \begin{array}{c} \exists \{(p_k,\mu_k)\} \subset \operatorname{epi}(a),\ \{(z_k,\omega_k)\} \subset \mathcal{P}_n \times \mathbb{R} \\ \text{with } (z_k,\omega_k) \in \widehat{N}_{\operatorname{epi}(a)}(p_k,\mu_k)\ \forall k, \\ \text{such that} \\ (p_k,\mu_k) \to (p,\mu) \text{ and } (z_k,\omega_k) \to (z,\omega) \end{array} \right\}. \tag{3.1}$$

By convention $N_{\mathrm{epi}\,(a)}(p,\mu) = \emptyset$ if $(p,\mu) \notin \mathrm{epi}\,(a)$. The abscissa mapping a is said to be subdifferentially regular at a point $(p,\mu) \in \mathrm{epi}\,(a)$ (equivalently, $\mathrm{epi}\,(a)$ is Clarke regular at (p,μ)) if

$$\widehat{N}_{\mathrm{epi}\,(a)}(p,\mu) = N_{\mathrm{epi}\,(a)}(p,\mu) \tag{3.2}$$

[RW98, Definition 7.25]. The goal of this section is to show that the set $\mathrm{epi}\,(a)$ is everywhere subdifferentially regular.

Some simplification in definition (3.1) is possible due to the continuity of a on its domain $\mathcal{M}_n$. Recall from (2.8) that

$$\widehat{N}_{\mathrm{epi}\,(a)}(p,\mu) = \left\{(\beta e_{(n,0)}, 0) \mid \beta \in \mathbb{C}\right\} \quad \text{whenever } \mu > a(p).$$

Since this subspace is constant on the set $\{(p,\mu) \mid \mu > a(p)\}$, we find that

$$N_{\mathrm{epi}\,(a)}(p,\mu) = \widehat{N}_{\mathrm{epi}\,(a)}(p,\mu) \quad \text{whenever } \mu > a(p).$$

Therefore, to establish that a is everywhere subdifferentially regular we need only establish the equivalence (3.2) at the points $(p, a(p))$ for $p \in \mathcal{M}_n$. In addition, from (2.7), we have $\left\{(\beta e_{(n,0)}, 0) \mid \beta \in \mathbb{C}\right\} \subset \widehat{N}_{\mathrm{epi}\,(a)}(p,\mu)$ for all $(p,\mu) \in \mathrm{epi}\,(a)$. Hence, it is always the case that

$$\widehat{N}_{\mathrm{epi}\,(a)}(p,\eta) \subset \widehat{N}_{\mathrm{epi}\,(a)}(p,\mu) \quad \text{whenever } a(p) \le \mu < \eta.$$

Therefore, the representation for the normal cone at the points $(p, a(p))$ for $p \in \mathcal{M}_n$ can be refined to

$$N_{\mathrm{epi}\,(a)}(p,a(p)) = \left\{ (z,\omega) \;\middle|\; \begin{array}{c} \exists\,\{p_k\} \subset \mathcal{M}_n,\ \{(z_k,\omega_k)\} \subset \mathcal{P}_n \times \mathbb{R} \\ \text{with } (z_k,\omega_k) \in \widehat{N}_{\mathrm{epi}\,(a)}(p_k, a(p_k))\ \forall\, k, \\ \text{such that} \\ p_k \to p \text{ and } (z_k,\omega_k) \to (z,\omega) \end{array} \right\}. \tag{3.3}$$

However, even with this simplification, we are confronted with a significant technical hurdle. Recall from Theorem 2.1 that the regular normals are characterized through the adjoint operator $\nabla F_p(0)^*$. Therefore, we now need to compute limits of these operators along sequences $p_k \to p$. But these adjoints are defined as linear transformations from $\mathcal{P}_n$ to $\mathcal{S}_{p_k}$ and are based on the inner products $\langle\cdot\,,\cdot\rangle_{p_k}$. How can we interpret limits of the adjoints $\nabla F_{p_k}(0)^*$ when the spaces $\mathcal{S}_{p_k}$ and their associated inner products $\langle\cdot\,,\cdot\rangle_{p_k}$ may not even be commensurate? The answer again lies with the local factorization lemma, Lemma 1.4.

Suppose that the polynomial $p \in \mathcal{M}_n$ has prime factorization (1.33) and let $\{p_k\}$ be a sequence of monic polynomials converging to p. Lemma 1.4 says that, by trimming off finitely many elements of the sequence if necessary, we may assume with no loss of generality that each of the polynomials p_k has a factorization of the form

$$p_k = \prod_{j=1}^{m} q_{kj}, \tag{3.4}$$

where the roots of the polynomials q_{kj}, $j = 1, \ldots, m$, are pairwise disjoint and $q_{kj} \to e_{(n_j,\lambda_j)}$ for each $j = 1, \ldots, m$. Moreover, since there are only finitely many partitions of n, we may assume with no loss in generality (by extracting a subsequence if necessary) that there exist positive integers ℓ_j, $j = 1, \ldots, m$, and n_{js}, $j = 1, \ldots, m$, $s = 1, \ldots, \ell_j$, with $\sum_{s=1}^{\ell_j} n_{js} = n_j$, such that, for each $k = 1, 2, \ldots$,

$$q_{kj} = \prod_{s=1}^{\ell_j} e_{(n_{js}, \lambda_{kjs})}, \tag{3.5}$$

where the complex numbers λ_{kjs}, $j = 1, \ldots, m$, $s = 1, \ldots, \ell_{kj}$ are distinct and satisfy $\lambda_{kjs} \to \lambda_j$ for $s = 1, \ldots, \ell_j$. Hence, for each $k = 1, 2, \ldots$, we have

$$\mathcal{S}_{p_k} = \mathbb{C} \times \left[\bigtimes_{j=1}^{m} \left(\mathcal{P}_{n_{j1}-1} \times \cdots \times \mathcal{P}_{n_{j\ell_j}-1} \right) \right], \tag{3.6}$$

$$F_{p_k}(v_0, v_{11}, \ldots, v_{1\ell_1}, \ldots, v_{m1}, \ldots, v_{m\ell_m}) = (1 + v_0) \prod_{j=1}^{m} \prod_{s=1}^{\ell_j} (e_{(n_{js}, \lambda_{kjs})} + v_{js}), \tag{3.7}$$

and

$$\nabla F_{p_k}(0)(h_0, h_{11}, \ldots, h_{1\ell_1}, \ldots, h_{m1}, \ldots, h_{m\ell_m}) = h_0 r_{k0} + \sum_{j=1}^{m} r_{kj} \left(\sum_{s=1}^{\ell_j} \hat{r}_{kjs} h_{js} \right), \tag{3.8}$$

where

$$r_{k0} = p_k \quad \text{and} \quad r_{kj_0} = \prod_{j \neq j_0} \prod_{s=1}^{\ell_j} e_{(n_{js}, \lambda_{kjs})}, \qquad j_0 = 1, \ldots, m,$$

and

$$\hat{r}_{kj_0 s_0} = \prod_{\substack{s=1 \\ s \neq s_0}}^{\ell_{j_0}} e_{(n_{j_0 s}, \lambda_{kj_0 s})}, \quad j_0 = 1, \ldots, m, \ s_0 = 1, \ldots, \ell_j \ .$$

Let us write $\hat{\mathcal{S}} = \mathcal{S}_{p_k}$, since $\mathcal{S}_{p_k}$ is fixed for all $k = 1, 2, \ldots$. Note that as $k \to \infty$, we have $r_{kj} \to r_j$, where r_j is defined in (1.37), for $j = 0, 1, \ldots, m$, and $\hat{r}_{kjs} \to e_{(\bar{n}_{js}, \lambda_j)}$, where $\bar{n}_{js} = n_j - n_{js}$, for $j = 1, \ldots, m$, $s = 1, \ldots, \ell_j$. Hence, $\nabla F_{p_k}(0) \to \Psi$, where the linear transformation $\Psi : \hat{\mathcal{S}} \to \mathcal{P}_n$ is given by

$$\Psi(h_0, h_{11}, \ldots, h_{1\ell_1}, \ldots, h_{m1}, \ldots, h_{m\ell_m}) = h_0 r_0 + \sum_{j=1}^{m} r_j \left(\sum_{s=1}^{\ell_j} e_{(\bar{n}_{js}, \lambda_j)} h_{js} \right). \tag{3.9}$$

Observe that the representation of $\nabla F_p(0)$ given in (1.36) and (1.37) enables us to write the operator Ψ as the composition

$$\Psi = \nabla F_p(0) \circ \Xi, \tag{3.10}$$

where the linear operator $\Xi : \hat{\mathcal{S}} \to \mathcal{S}_p$ is given by

$$\begin{aligned} &\Xi(h_0, h_{11}, \ldots, h_{1\ell_1}, \ldots, h_{m1}, \ldots, h_{m\ell_m}) \\ &\quad = \left(h_0, \sum_{s=1}^{\ell_1} e_{(\bar{n}_{1s}, \lambda_1)} h_{1s}, \ldots, \sum_{s=1}^{\ell_m} e_{(\bar{n}_{ms}, \lambda_m)} h_{ms} \right). \end{aligned} \tag{3.11}$$

Theorem 2.1 gives us access to the regular normals through the adjoint operators $\nabla F_p(0)^*$. Thus, in order to understand the normal cone, which consists of the limits of the regular normals, we need to come to an understanding of the limit of the adjoints $\nabla F_{p_k}(0)^*$. This limit is an adjoint of the operator Ψ. However, what this means needs clarification since each of the adjoints $\nabla F_{p_k}(0)^*$ arises from a different duality pairing. We need to determine the correct duality pairing for the definition of the adjoint Ψ^* so that it is the limit of the operators $\nabla F_{p_k}(0)^*$.

The duality pairing that we seek is obtained from our earlier observation (2.3) that the mapping $(p, q, \lambda) \mapsto \langle p\,, q\rangle_{(n,\lambda)}$ is continuous. This continuity implies that the pointwise limit of the inner products $\langle\cdot\,,\cdot\rangle_{p_k}$ exists as $k \to \infty$. Indeed, for each

$$u = (u_0, u_{11}, \dots, u_{1\ell_1}, \dots, u_{1\ell_m}, \dots, u_{m\ell_m})$$

and

$$v = (v_0, v_{11}, \dots, v_{1\ell_1}, \dots, v_{1\ell_m}, \dots, v_{m\ell_m})$$

in $\hat{\mathcal{S}}$, we have

$$\langle u\,, v\rangle_{p_k} \to \langle u\,, v\rangle_{\infty}\,,$$

where

$$\langle u\,, v\rangle_{\infty} = \langle u_0\,, v_0\rangle + \sum_{j=1}^{m}\sum_{s=1}^{\ell_j} \langle u_{js}\,, v_{js}\rangle_{(n_{js}-1,\lambda_j)}\ . \tag{3.12}$$

Therefore, if we define Ψ^* to be the adjoint of Ψ with respect to the duality pairings $(\mathcal{P}_n, \langle\cdot\,,\cdot\rangle)$ and $(\hat{\mathcal{S}}, \langle\cdot\,,\cdot\rangle_{\infty})$, then

$$\nabla F_{p_k}(0)^* \to \Psi^*. \tag{3.13}$$

Our next task is to derive a representation for the operator Ψ^*. Using the representation for Ψ given in (3.10), this reduces to deriving a representation for the adjoint of the operator Ξ. For this, the following lemma provides the key.

Lemma 3.1. *Let $\lambda_0 \in \mathbb{C}$ and let $\delta = (n_1, n_2, \dots, n_m)$ be a partition of n. Define $\mathcal{D}_\delta$ to be the product space*

$$\mathcal{D}_\delta = \mathcal{P}_{(n_1-1)} \times \mathcal{P}_{(n_2-1)} \times \cdots \times \mathcal{P}_{(n_m-1)},$$

endowed with the inner product

$$\langle (u_1, \dots, u_m)\,, (v_1, \dots, v_m)\rangle_{(\delta,\lambda_0)} = \sum_{j=1}^{m} \langle u_j\,, v_j\rangle_{(n_j-1,\lambda_0)}\,.$$

For $j = 1, 2, \dots, m$, define $\bar{n}_j = n - n_j$ and consider the linear transformation $\Phi_{(\delta,\lambda_0)} : \mathcal{D}_\delta \to \mathcal{P}_{n-1}$ given by

$$\Phi_{(\delta,\lambda_0)}(h_1, \dots, h_m) = \sum_{j=1}^{m} e_{(\bar{n}_j,\lambda_0)} h_j\ .$$

Then the adjoint of $\Phi_{(\delta,\lambda_0)}$ with respect to the duality pairings

$$(\mathcal{P}_{n-1}, \langle\cdot\,,\cdot\rangle_{(n-1,\lambda_0)}) \quad \textit{and} \quad (\mathcal{D}_\delta, \langle\cdot\,,\cdot\rangle_{(\delta,\lambda_0)})$$

is given by

$$\Phi^*_{(\delta,\lambda_0)}\left(\sum_{j=1}^{n} b_j e_{(n-j,\lambda_0)}\right) = \left(\sum_{j=1}^{n_1} b_j e_{(n_1-j,\lambda_0)}, \ldots, \sum_{j=1}^{n_m} b_j e_{(n_m-j,\lambda_0)}\right).$$

Proof. Define $J_s = \{j \mid n_j \geq s\}$ for $s = 1, 2, \ldots, n$. Note that J_s may be empty for some values of s. For example, if $m \geq 2$, then $J_n = \emptyset$. Let $(h_1, \ldots, h_m) \in \mathcal{D}_\delta$, where each $h_j \in \mathcal{P}_{n_j-1}$ has representation

$$h_j = a_{j1}e_{(n_j-1,\lambda_0)} + a_{j2}e_{(n_j-2,\lambda_0)} + \cdots + a_{jn_j} .$$

Given $q \in \mathcal{P}_{n-1}$ with

$$q = b_1 e_{(n-1,\lambda_0)} + \cdots + b_n,$$

we have

$$\begin{aligned}
&\left\langle q\,, \Phi_{(\delta,\lambda_0)}(h_1, \ldots, h_m)\right\rangle_{(n-1,\lambda_0)} \\
&= \left\langle \left(b_1 e_{(n-1,\lambda_0)} + \cdots + b_n\right)\,, \left(e_{(n-1,\lambda_0)}\left(\sum_{j\in J_1} a_{j1}\right)\right.\right. \\
&\qquad\qquad \left.\left. + \, e_{(n-2,\lambda_0)}\left(\sum_{j\in J_2} a_{j2}\right) + \cdots + \left(\sum_{j\in J_n} a_{jn}\right)\right)\right\rangle_{(n-1,\lambda_0)} \\
&= \operatorname{Re}\left[\bar{b}_1\left(\sum_{j\in J_1} a_{j1}\right) + \cdots + \bar{b}_n\left(\sum_{j\in J_n} a_{jn}\right)\right] \\
&= \operatorname{Re}\left[\sum_{s=1}^{n_1}\bar{b}_s a_{1s} + \sum_{s=1}^{n_2}\bar{b}_s a_{2s} + \cdots + \sum_{s=1}^{n_m}\bar{b}_s a_{ms}\right] \\
&= \left\langle \left(\sum_{s=1}^{n_1} b_s e_{(n_1-s,\lambda_0)}, \ldots, \sum_{s=1}^{n_m} b_s e_{(n_m-s,\lambda_0)}\right),\right. \\
&\qquad \left.\left(\sum_{s=1}^{n_1} a_{1s} e_{(n_1-s,\lambda_0)}, \ldots, \sum_{s=1}^{n_m} a_{ms} e_{(n_m-s,\lambda_0)}\right)\right\rangle_{(\delta,\lambda_0)} \\
&= \left\langle \left(\sum_{s=1}^{n_1} b_s e_{(n_1-s,\lambda_0)}, \ldots, \sum_{s=1}^{n_m} b_s e_{(n_m-s,\lambda_0)}\right), (h_1, \ldots, h_m)\right\rangle_{(\delta,\lambda_0)} .
\end{aligned}$$

Since this relation holds for all possible choices of $q \in \mathcal{P}_{n-1}$ and $(h_1, \ldots, h_m) \in \mathcal{D}_\delta$, we have established the result. ∎

By using the notation developed in Lemma 3.1, we can rewrite the operator $\Xi : \hat{\mathcal{S}} \to \mathcal{S}_p$, defined in (3.11), as

$$\Xi = \left(I, \Phi_{(\delta_1,\lambda_1)}, \ldots, \Phi_{(\delta_m,\lambda_m)}\right), \tag{3.14}$$

where $\delta_j = (n_{j1}, n_{j2}, \ldots, n_{j\ell_j})$ is a partition of n_j for each $j = 1, 2, \ldots, m$. Hence, from (3.10), we have

$$\Psi^* = \Xi^* \circ \nabla F_p(0)^*, \tag{3.15}$$

where $\Xi^* : \mathcal{S}_p \to \hat{\mathcal{S}}$ can be written as

$$\Xi^* = \left(I, \Phi^*_{(\delta_1,\lambda_1)}, \dots, \Phi^*_{(\delta_m,\lambda_m)}\right). \tag{3.16}$$

An explicit representation for the operator Ξ^* can be obtained by applying Lemma 3.1 to each of the operators $\Phi_{(\delta_j,\lambda_j)}$ for $j = 1, 2, \dots, m$.

We now prove the main result of this section.

THEOREM 3.2. *The abscissa mapping a is everywhere subdifferentially regular. Equivalently, epi(a) is Clarke regular.*

Proof. Let $p \in \mathcal{P}_n$ have factorization (1.33). Let $(z, \omega) \in N_{\mathrm{epi}\,(a)}(p, a(p))$ so that there exist sequences $\{p_k\} \subset \mathcal{M}_n$ and $\{(z_k, \omega_k)\} \subset \mathcal{P}_n \times \mathbb{R}$ such that $p_k \to p$, $(z_k, \omega_k) \to (z, \omega)$, and $(z_k, \omega_k) \in \widehat{N}_{\mathrm{epi}\,(a)}(p_k, a(p_k))$ for $k = 1, 2, \dots$. We need to show that $(z, \omega) \in \widehat{N}_{\mathrm{epi}\,(a)}(p, a(p))$.

The discussion preceding this theorem shows that we may assume with no loss of generality that (3.4)–(3.16) hold for the sequence $\{p_k\}$. Hence, we make free use of these facts and their associated notations.

Let $\tilde{\mathcal{I}}(p_k) = \{(j, s) \mid a(p_k) = \operatorname{Re} \lambda_{kjs}\}$. Since $(z_k, \omega_k) \in \widehat{N}_{\mathrm{epi}\,(a)}(p_k, a(p_k))$ for $k = 1, 2, \dots$, Theorem 2.1 states that $\omega_k \le 0$ and there exists

$$u_k = (u_{k0}, u_{k11}, \dots, u_{k1\ell_1}, \dots, u_{km1}, \dots, u_{km\ell_m}) \in \hat{\mathcal{S}}$$

with

$$u_{kjs} = \sum_{t=1}^{n_{js}} \mu_{kjst} e_{(n_{js}-t, \lambda_{kjs})}, \qquad \begin{array}{l} k = 1, 2, \dots, \\ j = 1, 2, \dots, m, \\ s = 1, 2, \dots, \ell_j\ , \end{array}$$

such that

$$u_k = \nabla F_{p_k}(0)^* z_k, \tag{3.17}$$

$$u_{kjs} = 0 \quad \text{for } (j, s) \notin \tilde{\mathcal{I}}(p_k), \tag{3.18}$$

$$\operatorname{Re} \mu_{kjs1} \le 0 \text{ and } \operatorname{Im} \mu_{kjs1} = 0 \quad \text{for } (j, s) \in \tilde{\mathcal{I}}(p_k), \tag{3.19}$$

$$\operatorname{Re} \mu_{kjs2} \le 0 \quad \text{for } (j, s) \in \tilde{\mathcal{I}}(p_k), \text{ and} \tag{3.20}$$

$$\sum_{(j,s)\in\tilde{\mathcal{I}}(p_k)} n_{js} \mu_{kjs1} = \omega_k. \tag{3.21}$$

Due to the finiteness of the index sets, we may assume with no loss of generality that $\tilde{\mathcal{I}}(p_k) = \tilde{\mathcal{I}}$ for all $k = 1, 2, \dots$. Define

$$\hat{\mathcal{I}} = \left\{ j \,\middle|\, (j, s) \in \tilde{\mathcal{I}} \text{ for some } s = 1, \dots, \ell_j \right\}.$$

By the continuity of the roots of the monic polynomials (as a multivalued mapping), we have $\hat{\mathcal{I}} \subset \mathcal{I}(p)$, where $\mathcal{I}(p)$ is defined in (1.38).

Using (3.13), let

$$u = \Psi^* z = \lim_{k\to\infty} \nabla F_{p_k}(0)^* z_k = \lim_{k\to\infty} u_k \tag{3.22}$$

and write

$$u = (u_0, u_{11}, \dots, u_{1\ell_1}, \dots, u_{m1}, \dots, u_{m\ell_m}) \in \hat{\mathcal{S}}$$

where

$$u_{js} = \sum_{t=1}^{n_{js}} \mu_{jst} e_{(n_{js}-t, \lambda_j)}, \tag{3.23}$$

with $\mu_{kjst} \to \mu_{jst}$ for $j = 1, \ldots, m,\ s = 1, \ldots, \ell_j,\ t = 1, \ldots, n_{js}$. By (3.22) and (3.17)–(3.21), we have

$$u_{js} = 0 \quad \text{for } (j, s) \notin \tilde{\mathcal{I}}, \tag{3.24}$$

$$\operatorname{Re} \mu_{js1} \le 0 \text{ and } \operatorname{Im} \mu_{js1} = 0 \quad \text{for } (j, s) \in \tilde{\mathcal{I}}, \tag{3.25}$$

$$\operatorname{Re} \mu_{js2} \le 0 \quad \text{for } (j, s) \in \tilde{\mathcal{I}}, \text{ and} \tag{3.26}$$

$$\sum_{(j,s) \in \tilde{\mathcal{I}}} n_{js} \mu_{js1} = \omega. \tag{3.27}$$

Set

$$(w_0, w_1, \ldots, w_m) = \nabla F_p(0)^* z, \tag{3.28}$$

with

$$w_j = \sum_{s=1}^{n_j} b_{js} e_{(n_j - s, \lambda_j)} \qquad \text{for } j = 1, \ldots, m. \tag{3.29}$$

By (3.15), (3.16), (3.22), and (3.28), we have

$$(u_{j1}, u_{j2}, \ldots, u_{j\ell_j}) = \Phi^*_{(\delta_j, \lambda_j)} w_j \qquad \text{for } j = 1, \ldots, m.$$

Consequently, by Lemma 3.1 and (3.23),

$$u_{js} = \sum_{t=1}^{n_{js}} \mu_{jst} e_{(n_{js}-t, \lambda_j)} = \sum_{t=1}^{n_{js}} b_{jt} e_{(n_{js}-t, \lambda_j)}$$

for $j = 1, \ldots, m,\ s = 1, \ldots, \ell_j$, or equivalently,

$$\mu_{jst} = b_{jt} \qquad \text{for } j = 1, \ldots, m,\ s = 1, \ldots, \ell_j,\ t = 1, \ldots, n_{js}. \tag{3.30}$$

Combining this with (3.24) and the definitions (3.23) and (3.29), we find

$$w_j = 0 \quad \text{for } j \notin \hat{\mathcal{I}}, \tag{3.31}$$

and combining (3.30) with (3.25) and (3.26) yields

$$\operatorname{Re} b_{j1} \le 0 \text{ and } \operatorname{Im} b_{j1} = 0 \quad \text{for } j \in \hat{\mathcal{I}} \text{ and} \tag{3.32}$$

$$\operatorname{Re} b_{j2} \le 0 \quad \text{for } j \in \hat{\mathcal{I}}. \tag{3.33}$$

Finally, by combining (3.30) with (3.27), we find

$$\sum_{(j,s) \in \tilde{\mathcal{I}}} n_{js} b_{j1} = \omega. \tag{3.34}$$

Note that the equivalence (3.30) implies that for every $j \in \hat{\mathcal{I}}$ for which $b_{j1} \neq 0$ it must be the case that $\mu_{js1} \neq 0$ for $s = 1, 2, \dots \ell_j$ and so $\{(j1), (j2), \dots, (j\ell_j)\} \subset \tilde{\mathcal{I}}$. Therefore, by (3.27) and (3.34),

$$\omega = \sum_{(j,s)\in\tilde{\mathcal{I}}} n_{js} b_{j1} = \sum_{j\in\hat{\mathcal{I}}} \sum_{s=1}^{\ell_j} b_{j1} n_{js} = \sum_{j\in\hat{\mathcal{I}}} b_{j1} n_j.$$

Consequently, by (3.28), (3.29), (3.31), (3.32), (3.33), the inclusion $\hat{\mathcal{I}} \subset \mathcal{I}(p)$, and Theorem 2.1, we find that $(z, \omega) \in \widehat{N}_{\mathrm{epi}\,(a)}(p, a(p))$, which establishes the result. $\square$

Just as the set of normal vectors is defined to be the set of limits of regular normal vectors, the set of subgradients is defined to be the set of limits of regular subgradients:

$$\partial a(p) = \left\{ q \;\middle|\; \begin{array}{c} \exists\, \{p_k\} \subset \mathrm{dom}\,(a) \text{ and } \{q_k\} \subset \mathcal{P}_n, \\ \text{such that } q_k \in \hat{\partial} a(p_k)\, \forall\, k = 1, 2, \dots, \\ p_k \to p \text{ and } q_k \to q \end{array} \right\}, \tag{3.35}$$

with $\partial a(p) = \emptyset$ if $p \notin \mathrm{dom}\, a = \mathcal{M}_n$. The set of *horizon subgradients*, denoted $\partial^\infty \alpha(p)$, is defined similarly, however, instead of $q_k \to q$ one has $t_k q_k \to q$ for some sequence of positive real numbers $\{t_k\}$ converging to zero. By convention, we have $\partial^\infty a(p) = \{0\}$ if $p \notin \mathrm{dom}\, a$. As in the case of regular subgradients, there is a relationship between these subgradients and the normal cone at a polynomial $p \in \mathcal{M}_n$ [RW98, Theorem 8.9]:

$$\partial a(p) = \left\{ q \;\middle|\; (q, -1) \in N_{\mathrm{epi}\,(a)}(p, a(p)) \right\}$$

and

$$\partial^\infty a(p) = \left\{ q \;\middle|\; (q, 0) \in N_{\mathrm{epi}\,(a)}(p, a(p)) \right\}.$$

Using these relationships, Theorem 3.2 and [RW98, Corollary 8.11] imply that

$$\partial a(p) = \hat{\partial} a(p) \quad \text{and} \quad \partial^\infty a(p) = \hat{\partial} a(p)^\infty \tag{3.36}$$

whenever $p \in \mathcal{M}_n$ (see (2.27) and (2.28)).

The subdifferential regularity of the abscissa mapping implies that it possesses a rich subdifferential calculus. For example, the following chain rule holds.

THEOREM 3.3 (see [RW98, Theorem 10.6]). *Let X be a finite dimensional Euclidean space, and suppose $G : X \to \mathcal{P}_n$ is continuously differentiable in the real sense. Consider the composite mapping $g = a \circ G$. If $x \in X$ is such that $G(x) \in \mathcal{M}_n$ and the only polynomial $q \in \partial^\infty a(G(x))$ with $\nabla G(x)^* q = 0$ is $q = 0$, then*

$$\partial g(x) = \nabla G(x)^* \partial a(G(x)), \qquad \partial^\infty g(x) = \nabla G(x)^* \partial^\infty a(G(x)),$$

and

$$dg(x)(d) = da(G(x))(\nabla G(x) d).$$

To illustrate these results, we apply Theorem 3.3 to the example studied in [BLO]. Let X be $\mathbb{C}^n$ with the standard real inner product, and consider the composition of the abscissa function with the affine mapping $G : \mathbb{C}^n \to \mathcal{P}_n$ given by

$$G(x) = (1 + x_0) e_{(n,0)} + x_1 e_{(n-1,0)} - \sum_{j=2}^{n} x_{j-1} e_{(n-j,0)}.$$

In [BLO, Theorem 2.1], it is shown that $x = 0$ is a strict global minimizer of the function $g = a \circ G$. Since G is affine, we have

$$\nabla G(0)d = d_0 e_{(n,0)} + d_1 e_{(n-1,0)} - \sum_{j=2}^{n} d_{j-1} e_{(n-j,0)},$$

and

$$\nabla G(0)^* \sum_{j=0}^{n} y_j e_{(n-j,0)} = (y_0, y_1 - y_2, -y_3, \dots, -y_n).$$

The representation for $\partial^\infty a(e_{(n,0)})$ given by (3.36) and (2.26) shows that the only $q \in \partial^\infty a(e_{(n,0)})$ with $\nabla G(0)^* q = 0$ is $q = 0$. Therefore, Theorem 3.3 and the relations (3.36) and (2.25) show that

$$\hat{\partial} g(0) = \partial g(0) = \left\{ \left(z_0, z_1 - \frac{1}{n}, z_2, \dots, z_{n-1} \right) \mid \operatorname{Re} z_1 \geq 0 \right\}.$$

Finally, observe that since the origin is in the interior of $\hat{\partial} g(0)$, we have, directly from the definition of regular subgradients, that $x = 0$ is a *sharp* minimizer of g in the sense that there exist $\epsilon > 0$ and $\kappa > 0$ such that

$$g(x) \geq g(0) + \kappa \|x\| \qquad \text{whenever} \quad \|x\| \leq \epsilon.$$

Further consequences of these results are explored in [BLO].

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REFERENCES

[BLO] J.V. Burke, A.S. Lewis, and M.L. Overton, *Optimizing matrix stability*, Proc. Amer. Math. Soc., to appear.

[BO] J.V. Burke and M.L. Overton, *Variational analysis of non-Lipschitz spectral functions*, Math. Programming, to appear.

[Lev80] L.V. Levantovskii, *On singularities of the boundary of a stability region*, Moscow Univ. Math. Bull., 35 (1980), pp. 19–22.

[RW98] R.T. Rockafellar and R. J. B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.