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PERTURBING THE CRITICALLY DAMPED WAVE EQUATION*

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Abstract. We consider the wave equation with viscous damping. The equation is said to be critically damped when the damping is that value for which the spectral abscissa of the associated wave operator is minimized within the class of constant dampings. The critically damped wave operator possesses a nonsemisimple eigenvalue. We present a detailed study of the splitting of this eigenvalue under bounded perturbations of the damping and subsequently show that the critical choice is a local minimizer of the spectral abscissa over lines in the class of all bounded dampings.

Key words. nonselfadjoint operator, spectral abscissa, multiple eigenvalue

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1. Introduction.

We are concerned with the damped wave equation

$$u_{tt}(x,t) - \Delta u(x,t) + 2a(x)u_t(x,t) = 0, \quad u(\cdot,t) \in H_0^1(\Omega), \tag{1.1}$$

on the open bounded connected set $\Omega \subset \mathbf{R}^d$, where a is near critical. We define our use of the word critical by analogy to the scalar damped linear oscillator

$$y''(t) + y(t) + 2ay'(t) = 0. \tag{1.2}$$

One notes that $Y(t) = [y(t)\ y'(t)]^T$ satisfies the first-order system $Y'(t) = A(a)Y(t)$, where

$$A(a) = \begin{pmatrix} 0 & 1 \\ -1 & -2a \end{pmatrix},$$

and solves the corresponding initial value problem in terms of the eigenvalues and eigenvectors of $A(a)$. The effectiveness of the damping is measured by the decay rate

$$\omega(a) \equiv \inf\{\alpha : \exists C \text{ s.t. } \|Y(t)\|^2 \le C\|Y(0)\|^2 e^{2\alpha t},\ \forall Y(0),\ \forall t \ge 0\}.$$

In this context one easily identifies $\omega(a)$ with the spectral abscissa of $A(a)$, i.e., with

$$\mu(a) \equiv \sup_{\lambda \in \sigma(a)} \Re\lambda,$$

where $\sigma(a)$ is the spectrum of $A(a)$. As $\mu(a) = -a + \Re\sqrt{a^2 - 1}$ achieves its minimum at $a = 1$, (1.2) is said to be over(under)damped if $a > 1$ ($a < 1$) and critically damped if $a = 1$.

Analogously, with $v(t) = [u(t)\ u_t(t)]$ we interpret (1.1) as $v_t = A(a)v$, where

$$A(a) = \begin{pmatrix} 0 & I \\ \Delta & -2a \end{pmatrix}, \qquad D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \tag{1.3}$$

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is densely defined in the Hilbert space $X = H_0^1(\Omega) \times L^2(\Omega)$ with inner product

$$\langle [f,g],[u,v]\rangle = \int_\Omega \nabla f \cdot \nabla \overline{u} + g\overline{v}\,dx.$$

When a lies in $L^\infty(\Omega)$, this $A(a)$ has a compact inverse and so a discrete spectrum, $\sigma(a)$. Cox and Zuazua [3] have shown that one may identify the decay rate with the spectral abscissa when $d = 1$ and a is of bounded variation. When $d > 1$ the spectral abscissa is known to be insufficient. Lebeau [6] has shown in this case that it is sufficient to integrate a along the generalized geodesics of Ω. In particular, if $a \in C^\infty(\overline{\Omega})$ then

$$\omega(a) = \max\{\mu(a), \gamma(a)\},$$

where, denoting by G the generalized geodesics on Ω,

$$\gamma(a) \equiv -\lim_{t\to\infty} \inf_{g\in G} \frac{1}{t}\int_0^t a(g(s))\,ds.$$

When a is constant we find $\gamma(a) = -a$. To evaluate $\mu(a)$ in the constant case we must compute the eigenvalues of the operator $A(a)$ defined in (1.3). We express these in terms of

$$0 < \Lambda_1 < \Lambda_2 \le \Lambda_3 \le \cdots \to \infty,$$

the eigenvalues, repeated according to their multiplicity, of $-\Delta$ on $H_0^1(\Omega)$. In particular, the eigenvalues of $A(a)$ are

$$\lambda_{\pm n} = -a \pm \sqrt{a^2 - \Lambda_n}, \quad n = 1, 2, \ldots. \tag{1.4}$$

We illustrate this formula in Fig. 1 by tracing the non-Lipschitz coalescence of λ_1 and λ_{-1} when $\Omega = (0,1)$ and $\Lambda_1 = \pi^2$.

As Λ_1 is simple it follows that $\lambda_1(\sqrt{\Lambda_1}) = \lambda_{-1}(\sqrt{\Lambda_1}) = -\sqrt{\Lambda_1}$ has algebraic multiplicity two and geometric multiplicity one. For constant a, then,

$$\omega(a) = \mu(a) = -a + \Re\sqrt{a^2 - \Lambda_1}.$$

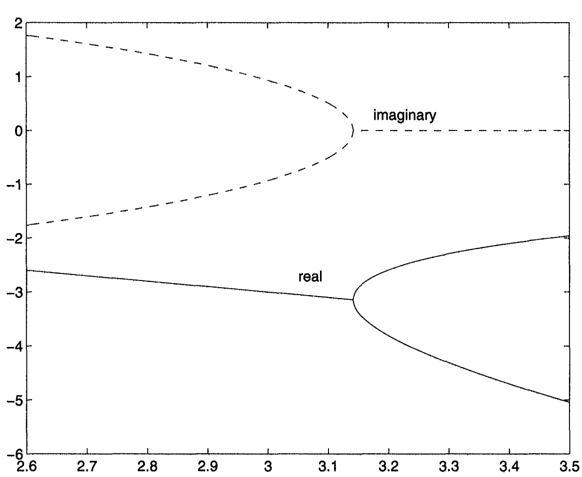

FIG. 1. *The real and imaginary parts of $\lambda_{\pm 1}(a)$ for a near π when $\Omega = (0,1)$.*

As $a = \sqrt{\Lambda_1}$ is the global minimizer of ω over $a \in \mathbf{R}$ it is by analogy to (1.2) that we deem (1.1) critically damped when $a = \sqrt{\Lambda_1}$. We conjecture that $a = \sqrt{\Lambda_1}$ in fact minimizes ω over $a \in L^\infty(\Omega)$. In this note we provide a local validation of this conjecture. More precisely, for fixed $a_1 \in L^\infty(\Omega)$, we show that $\mu(\varepsilon)$, the spectral abscissa of

$$A(\varepsilon) = A_0 + \varepsilon A_1 = \begin{pmatrix} 0 & I \\ \Delta & -2\sqrt{\Lambda_1} \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & -2a_1 \end{pmatrix},$$

is increasing for $0 < \varepsilon < \delta(a_1)$. The upshot of this result is that in each $L^\infty(\Omega)$ neighborhood of $\sqrt{\Lambda_1}$ there exists an a and a choice of initial data such that the solution of (1.1) corresponding to a decays slower than that corresponding to $\sqrt{\Lambda_1}$.

This A_0 is the critically damped wave operator. In §2 we compile the raw data required by the perturbation theory, i.e., the biorthogonal set of root vectors of A_0 and its adjoint and the Laurent expansion of the resolvent. In §3 we invoke Kato [4] in calculating the weighted averages and the individual branches of the multiple eigenvalues of $A(\varepsilon)$. We find that, independent of the choice of $a_1 \in L^\infty(\Omega)$, the eigenvalue λ_1 splits and that at least one of its branches travels to the right.

2. The critically damped wave operator. We denote by $\{\phi_n\}_{n=1}^\infty$ the orthonormal (in $L^2(\Omega)$) basis of eigenfunctions of $-\Delta$ associated with $\{\Lambda_n\}_{n=1}^\infty$. Let

$$\{n_k\}_{k=1}^\infty = \{1\} \cup \{j : \Lambda_{j-1} < \Lambda_j,\ j \geq 2\}$$

be that increasing sequence that indexes the distinct elements of $\{\Lambda_n\}_{n=1}^\infty$. The difference $m_k \equiv n_{k+1} - n_k$ is simply the multiplicity of Λ_{n_k}. The orthogonal projection onto the corresponding m_k-dimensional eigenspace is denoted by

$$p_k = \sum_{j=n_k}^{n_k+m_k-1} (\cdot, \phi_j)\phi_j.$$

Here $(\cdot,\cdot)$ denotes the standard inner product on $L^2(\Omega)$. We shall use $\|\cdot\|_2$ to indicate the associated norm. It follows from (1.4) that the eigenvalues of A_0 are

$$\lambda_{\pm n} = -\sqrt{\Lambda_1} \pm i\sqrt{\Lambda_n - \Lambda_1}, \quad n = 1, 2, \ldots.$$

As A_0 is real these are also the eigenvalues of its adjoint

$$A_0^* = \begin{pmatrix} 0 & -I \\ -\Delta & -2\sqrt{\Lambda_1} \end{pmatrix}, \quad D(A_0^*) = D(A_0).$$

We record the corresponding biorthogonal system of root vectors.

PROPOSITION 2.1. *The geometric multiplicity of λ_1 is one. The algebraic multiplicity of λ_1 is two. For $n \neq 1$ the algebraic multiplicity of λ_n coincides with its geometric multiplicity. This value is the multiplicity of $\Lambda_{|n|}$. The vectors*

$$v_1 = \frac{\phi_1}{\sqrt{\Lambda_1}}[1 \quad -\sqrt{\Lambda_1}], \quad v_{-1} = \frac{\phi_1}{\sqrt{\Lambda_1}}[0 \quad 1],$$

$$v_{\pm n} = \frac{\phi_n}{\sqrt{\Lambda_n - \lambda_{\pm n}^2}}[1 \quad \lambda_{\pm n}], \quad n = 2, 3, \ldots,$$

$$w_{-1} = \phi_1[1 \quad \sqrt{\Lambda_1}], \quad w_1 = \frac{\phi_1}{\sqrt{\Lambda_1}}[1 \quad 0],$$

$$w_{\pm n} = \frac{\phi_n}{\overline{\sqrt{\Lambda_n - \lambda_{\pm n}^2}}}[1 \quad -\lambda_{\mp n}], \quad n = 2, 3, \ldots$$

satisfy

$$A_0 v_1 = \lambda_1 v_1, \quad (A_0 - \lambda_1) v_{-1} = v_1,$$
$$A_0^* w_{-1} = \lambda_1 w_{-1}, \quad (A_0^* - \lambda_1) w_1 = w_{-1},$$
$$A_0 v_{\pm n} = \lambda_{\pm n} v_{\pm n}, \quad A_0^* w_{\pm n} = \lambda_{\mp n} w_{\pm n}, \quad n = 2, 3, \ldots$$

and

$$\langle v_i, w_j \rangle = \delta_{ij}.$$

Proof. As A_0 and A_0^* are constant coefficient operators, each of our claims follows on direct verification. □

The perturbation theory of the next section is predicated on the fact that the λ_n are poles of the resolvent $R(z) = (A_0 - z)^{-1}$. We follow Kato [4, § III.6.5] and develop its Laurent expansion about λ_1. Namely,

$$R(z) = -\frac{P_1}{z - \lambda_1} - \frac{D_1}{(z - \lambda_1)^2} + \sum_{n=0}^{\infty} (z - \lambda_1)^n S_1^{n+1},$$

where

$$P_1 = \langle \cdot, w_1 \rangle v_1 + \langle \cdot, w_{-1} \rangle v_{-1}$$

is the eigenprojection of X onto the invariant subspace spanned by v_1 and v_{-1}, D_1 is the corresponding eigennilpotent

$$D_1 = (A_0 - \lambda_1) P_1 = \langle \cdot, w_{-1} \rangle v_1,$$

and S_1, the associated reduced resolvent, satisfies

$$(A_0 - \lambda_1) S_1 = I - P_1 \quad \text{and} \quad S_1 P_1 = P_1 S_1 = 0. \tag{2.1}$$

Regarding the remaining eigenvalues we need only collect the eigenprojections:

$$P_{\pm k} \equiv \sum_{j = \pm n_k}^{\pm (n_k + m_k - 1)} \langle \cdot, w_j \rangle v_j, \quad k > 1.$$

That $\{v_{\pm n}\}_{n=1}^{\infty}$ comprises a basis for X follows from the simple calculation

$$\begin{aligned} P_1[f, g] &= [p_1 f, p_1 g] \qquad \forall\, [f, g] \in X, \\ (P_k + P_{-k})[f, g] &= [p_k f, p_k g], \quad k > 1, \qquad \forall\, [f, g] \in X. \end{aligned} \tag{2.2}$$

This will permit us to represent S_1. For convenience, we state this in the form of a Fredholm Alternative. It follows from [4, Thm. IV.5.28] that $B \equiv (A_0 - \lambda_1)$ is Fredholm and that the range of B is the orthogonal complement of the null space of the adjoint of B. Recall that w_{-1} spans the null space of B^*.

LEMMA 2.2. *Given $\xi \in X$, the equation $B\psi = \xi$ possesses a solution if and only if $\langle \xi, w_{-1} \rangle = 0$. If, in addition, one specifies $\langle \psi, v_1 \rangle = 0$, then the unique solution is*

$$\psi = \langle \xi, w_1 \rangle v_{-1} + i \sum_{k=2}^{\infty} \frac{(P_{-k} - P_k)\xi}{\sqrt{\Lambda_{n_k}} - \Lambda_1}.$$

Proof. We expand ψ and ξ in terms of

$$\psi = \langle \psi, w_{-1} \rangle v_{-1} + \sum_{k=2}^{\infty} (P_k + P_{-k})\psi, \tag{2.3}$$

$$\xi = \langle \xi, w_1 \rangle v_1 + \sum_{k=2}^{\infty} (P_k + P_{-k})\xi.$$

Equating $B\psi$ and ξ we find

$$\langle \psi, w_{-1} \rangle v_1 + \sum_{k=2}^{\infty} \{(\lambda_{n_k} - \lambda_1) P_k + (\lambda_{-n_k} - \lambda_1) P_{-k}\}\psi = \langle \xi, w_1 \rangle v_1 + \sum_{k=2}^{\infty} (P_k + P_{-k})\xi.$$

As $P_j P_k = \delta_{jk} P_j$ it now follows that $\langle \psi, w_{-1} \rangle = \langle \xi, w_1 \rangle$ and, for $k > 1$,

$$P_{\pm k}\psi = \frac{P_{\pm k}\xi}{\lambda_{\pm n_k} - \lambda_1} = \frac{P_{\pm k}\xi}{\pm i\sqrt{\Lambda_{n_k} - \Lambda_1}}.$$

Substituting these into (2.3) yields the desired result. □

We remark that this ψ has the simple representation

$$\psi = (D_1^+ + S_1)\xi, \tag{2.4}$$

where

$$S_1 = i \sum_{k=2}^{\infty} \frac{P_{-k} - P_k}{\sqrt{\Lambda_{n_k} - \Lambda_1}} \tag{2.5}$$

satisfies (2.1) and

$$D_1^+ = \langle \cdot, w_1 \rangle v_{-1}$$

is the pseudoinverse of D_1.

3. The perturbed wave operator. We note that $R(z)$ is well defined for z on a small circle about λ_1, e.g.,

$$\Gamma \equiv \{-\sqrt{\Lambda_1} + \tfrac{1}{2} e^{i\theta} \sqrt{\Lambda_2 - \Lambda_1} : 0 \le \theta < 2\pi\}.$$

Our first task is to show that, for sufficiently small ε, the resolvent of $A(\varepsilon)$ is also well defined on Γ.

We work within the context of holomorphic families of type (A). That $A(\varepsilon)$ is indeed of this type (see Kato [4, Rem. VII.2.7]) will follow from the existence of two constants α and β for which

$$\|A_1 u\|_X \le \alpha \|u\|_X + \beta \|A_0 u\|_X \qquad \forall\, u \in D(A_0).$$

In terms of $u = [f, g]$ this criterion takes the form

$$\|2a_1 g\|_2 \le \alpha \|g\|_2 + \beta \|\nabla g\|_2 \qquad \forall\, g \in H_0^1(\Omega).$$

The most obvious choice is $\alpha = 2\|a_1\|_\infty$ and $\beta = 0$. An alternate choice presents itself in the one-dimensional case. In particular, if $\Omega = (0, \ell)$ and $g \in H_0^1(\Omega)$, then from

$$g(x) = \int_0^x g'(y)\,dy \quad \text{it follows that} \quad \|g\|_\infty \le \sqrt{\ell}\|g'\|_2.$$

As a result, one may choose $\alpha = 0$ and $\beta = 2\sqrt{\ell}\|a_1\|_2$ in this case.

Having chosen α and β we now deduce from [4, Rem. VII.2.9] that $R(z,\varepsilon) \equiv (A(\varepsilon) - z)^{-1}$ exists for $z \in \Gamma$ when

$$|\varepsilon| < r \equiv \min_{z\in\Gamma}(\alpha\|R(z)\|_{L(X)} + \beta\|A_0 R(z)\|_{L(X)})^{-1}. \tag{3.1}$$

Here $\|\cdot\|_{L(X)}$ denotes the operator norm on the space of bounded linear operators on X. It follows, for $|\varepsilon| < r$, that $A(\varepsilon)$ has as many eigenvalues in Γ as does A_0 and that their average within Γ is holomorphic. We denote by $\lambda_{\pm 1}(\varepsilon)$ the two eigenvalues of $A(\varepsilon)$ contained in Γ. We follow [4, §II.2.2] in calculating the power series representation of their average. Namely,

$$\begin{aligned} \hat{\lambda}_1(\varepsilon) &\equiv \frac{\lambda_1(\varepsilon) + \lambda_{-1}(\varepsilon)}{2} \\ &= \lambda_1 + \frac{\varepsilon}{2}\mathrm{tr}\,(A_1P_1) - \frac{\varepsilon^2}{2}\mathrm{tr}\,(A_1S_1^2A_1D_1 + A_1S_1A_1P_1) + O(\varepsilon^3). \end{aligned} \tag{3.2}$$

Having assembled the necessary operators in the previous section, we find the following result.

PROPOSITION 3.1. *For $|\varepsilon| < r$ there holds*

$$\hat{\lambda}_1(\varepsilon) = \lambda_1 - \varepsilon(a_1\phi_1, \phi_1) - 4\varepsilon^2\sqrt{\Lambda_1}\sum_{n=2}^{\infty}\frac{(a_1\phi_1, \phi_n)^2}{\Lambda_n - \Lambda_1} + O(\varepsilon^3).$$

Proof. We compute the traces in (3.2). These are routine calculations, see [4, §III.4.3], because each of the required operators is of rank at most two. We make use of the fact that A_1 is selfadjoint and that its range is orthogonal to w_1. In particular,

$$\mathrm{tr}\,A_1P_1 = \langle A_1v_1, w_1\rangle + \langle A_1v_{-1}, w_{-1}\rangle = -2(a_1\phi_1, \phi_1).$$

Recalling (2.5) we find

$$S_1^2 = -\sum_{k=2}^{\infty}\frac{P_k + P_{-k}}{\Lambda_{n_k} - \Lambda_1},$$

and proceed to compute

$$\begin{aligned} \mathrm{tr}\,(A_1S_1^2A_1D_1) &= \langle A_1S_1^2A_1v_1, w_{-1}\rangle \\ &= \langle S_1^2A_1v_1, A_1w_{-1}\rangle \\ &= -\sum_{k=2}^{\infty}\frac{\langle (P_k + P_{-k})A_1v_1, A_1w_{-1}\rangle}{\Lambda_{n_k} - \Lambda_1} \\ &= 4\sqrt{\Lambda_1}\sum_{k=2}^{\infty}\frac{(p_k(a_1\phi_1), a_1\phi_1)}{\Lambda_{n_k} - \Lambda_1} \\ &= 4\sqrt{\Lambda_1}\sum_{n=2}^{\infty}\frac{(a_1\phi_1, \phi_n)^2}{\Lambda_n - \Lambda_1}. \end{aligned}$$

The fourth line follows from (2.2). Similarly,

$$
\begin{aligned}
\operatorname{tr}(A_1S_1A_1P_1) &= \langle A_1S_1A_1v_1, w_1\rangle + \langle A_1S_1A_1v_{-1}, w_{-1}\rangle \\
&= \langle S_1A_1v_{-1}, A_1w_{-1}\rangle \\
&= i\sum_{k=2}^{\infty} \frac{\langle (P_{-k}-P_k)A_1v_{-1}, A_1w_{-1}\rangle}{\sqrt{\Lambda_{n_k}-\Lambda_1}} \\
&= i\sum_{n=2}^{\infty} \frac{\langle A_1v_{-1}, w_{-n}\rangle\langle v_{-n}, A_1w_{-1}\rangle - \langle A_1v_{-1}, w_n\rangle\langle v_n, A_1w_{-1}\rangle}{\sqrt{\Lambda_n-\Lambda_1}} \\
&= -8\sum_{n=2}^{\infty} \frac{(a_1\phi_1,\phi_n)^2}{\sqrt{\Lambda_n-\Lambda_1}}\Im\frac{\lambda_n^2}{\Lambda_n-\lambda_n^2} \\
&= 4\sqrt{\Lambda_1}\sum_{n=2}^{\infty} \frac{(a_1\phi_1,\phi_n)^2}{\Lambda_n-\Lambda_1}.
\end{aligned}
$$

This now yields the stated expansion. □

Though one may just as easily compute additional terms, we note that the coefficients of ε and ε^2 cannot vanish simultaneously for nontrivial a_1. Kato [4, §II.3.1] also permits us to bound the magnitude of the remainder term, denoted $O(\varepsilon^3)$ above, by

$$\frac{1}{2}\sqrt{\Lambda_2-\Lambda_1}\frac{|\varepsilon|^3}{r^2(r-|\varepsilon|)}.$$

With regard to our stated goal of minimizing μ we observe that this proposition states that $\hat{\lambda}(\varepsilon)$ increases with ε when $(a_1\phi_1,\phi_1)<0$. For the average to increase one of the summands must increase and hence μ must so increase. We still must show that this is also the case when $(a_1\phi_1,\phi_1)\geq 0$. This will require a careful study of the splitting of λ_1.

Though Kato [4,§VII.1.3] permits us to conclude that $\lambda_{\pm 1}(\varepsilon)$ are branches of an analytic function with at worst an algebraic singularity at $\varepsilon=0$, he refers to Baumgartel [1] for the relevant expansion results. We shall require very little of this highly technical general theory. For, in considering a linear perturbation of a geometrically simple eigenvalue, we shall be able, as above, to compute the required coefficients by hand. The process we adopt is termed "calculation by recursion" by Baumgartel [1, §7.4.13] and the "method of undetermined coefficients" by Vainberg and Trenogin [7, §32.5]. It goes back to Vishik and Lyusternik [8] and, in our context, amounts to no more than repeated application of Lemma 2.2.

In particular, to solve

$$A(\varepsilon)y(\varepsilon)=\lambda(\varepsilon)y(\varepsilon), \quad \lambda(0)=\lambda_1, \tag{3.3}$$

is to solve

$$By(\varepsilon)=(\zeta(\varepsilon)-\varepsilon A_1)y(\varepsilon), \quad \zeta(0)=0, \tag{3.4}$$

where

$$B\equiv A_0-\lambda_1 \quad \text{and} \quad \zeta(\varepsilon)=\lambda(\varepsilon)-\lambda_1.$$

Recall that Lemma 2.2 addresses the solvability of (3.4). The starting point is the pair of Puiseux series (see, e.g., [4, §II.1.2]),

$$\zeta(\varepsilon)=\sum_{k=1}^{\infty}\zeta_k\varepsilon^{k/2}, \quad y(\varepsilon)=\sum_{k=0}^{\infty}y_k\varepsilon^{k/2},$$

convergent for $|\varepsilon| < r$. We now simply insert these into (3.3) and equate like powers of ε. As above we proceed until we arrive at a nonzero coefficient.

ε^0: We find $By_0 = 0$ and so $y_0 = v_1$.

$\varepsilon^{1/2}$: We find $By_1 = \zeta_1 y_0$ and so $y_1 = \zeta_1 v_{-1}$.

ε: We find $By_2 = \zeta_2 y_0 + \zeta_1 y_1 - A_1 y_0 = (\zeta_2 - A_1)v_1 + \zeta_1^2 v_{-1}$. This equation is solvable precisely when the right side is orthogonal to w_{-1}, that is, when $\langle(\zeta_2 - A_1)v_1 + \zeta_1^2 v_{-1}, w_{-1}\rangle = 0$. As $\langle v_i, w_j\rangle = \delta_{ij}$, this condition reads simply

$$\zeta_1^2 = \langle A_1 v_1, w_{-1}\rangle = 2\sqrt{\Lambda_1}(a_1\phi_1, \phi_1).$$

As a result, we find

$$\lambda_{\pm 1} = \lambda_1 \pm \left(2\sqrt{\Lambda_1}(a_1\phi_1, \phi_1)\varepsilon\right)^{1/2} + O(\varepsilon), \tag{3.5}$$

and so, when $(a_1\phi_1, \phi_1) > 0$, we see that $\lambda_1(\varepsilon) > \lambda_1$ and so μ is increasing. To determine the direction of μ when $(a_1\phi_1, \phi_1) = 0$ we must find (at least) ζ_2. We first represent y_2 in the manner used in (2.4), i.e.,

$$\begin{aligned} y_2 &= (D_1^+ + S_1)((\zeta_2 - A_1)v_1 + \zeta_1^2 v_{-1}) \\ &= (\zeta_2 - \langle A_1 v_1, w_1\rangle)v_{-1} - S_1 A_1 v_1 \\ &= \zeta_2 v_{-1} - S_1 A_1 v_1. \end{aligned}$$

$\varepsilon^{3/2}$: We find $By_3 = \zeta_3 y_0 + \zeta_2 y_1 + \zeta_1 y_2 - A_1 y_1$. As above, this right side must be orthogonal to w_{-1}. That is, recalling y_0 and y_1,

$$\zeta_1(\zeta_2 + \langle y_2 - A_1 v_{-1}, w_{-1}\rangle) = 0.$$

Now recalling y_2 we find

$$\zeta_1(\zeta_2 + \langle \zeta_2 v_{-1} - S_1 A_1 v_1 - A_1 v_{-1}, w_{-1}\rangle) = 0.$$

That is,

$$\zeta_1(2\zeta_2 + 2(a_1\phi_1, \phi_1)) = 0.$$

Hence, so long as $\zeta_1 \neq 0$ we find

$$\zeta_2 = -(a_1\phi_1, \phi_1). \tag{3.6}$$

Note that this permits us to recover, to first order, the previous proposition. Namely, $\hat{\lambda}_1(\varepsilon) = \lambda_1 - (a_1\phi_1, \phi_1)\varepsilon + O(\varepsilon^2)$. Equation (3.6), however, is only a partial result. To determine ζ_2 when $\zeta_1 = 0$ we must proceed to (at least) the next level. We first represent

$$\begin{aligned} y_3 &= (D_1^+ + S_1)((\zeta_3 - \zeta_1 S_1 A_1)v_1 + \zeta_1(2\zeta_2 - A_1)v_{-1}) \\ &= \zeta_3 v_{-1} - \zeta_1 S_1^2 A_1 v_1 - \zeta_1 S_1 A_1 v_{-1}. \end{aligned}$$

ε^2: We find $By_4 = \zeta_4 y_0 + \zeta_3 y_1 + \zeta_2 y_2 + \zeta_1 y_3 - A_1 y_2$. The solvability condition now takes the form

$$2\zeta_1\zeta_3 + \langle(\zeta_2 - A_1)(\zeta_2 v_{-1} - S_1 A_1 v_1), w_{-1}\rangle = 0.$$

This simplifies to

$$2\zeta_1\zeta_3 + \zeta_2^2 - \zeta_2\langle A_1 v_{-1}, w_{-1}\rangle + \langle A_1 S_1 A_1 v_1, w_{-1}\rangle = 0,$$

and so, if $\zeta_1 = 0$, we find, on recalling the proof of Proposition 3.1, that

$$\begin{aligned}\zeta_2^2 &= -\langle A_1 S_1 A_1 v_1, w_{-1}\rangle \\ &= \sqrt{\Lambda_1}\langle A_1 S_1 A_1 v_{-1}, w_{-1}\rangle \\ &= \sqrt{\Lambda_1}\,\mathrm{tr}\,(A_1 S_1 A_1 P_1) \\ &= 4\Lambda_1 \sum_{n=2}^{\infty} \frac{(a_1\phi_n, \phi_n)^2}{\Lambda_n - \Lambda_1}.\end{aligned}$$

As a result, if $\zeta_1 = 0$ then

$$\lambda_{\pm 1}(\varepsilon) = \lambda_1 \pm \varepsilon \left(4\Lambda_1 \sum_{n=2}^{\infty} \frac{(a_1\phi_n, \phi_n)^2}{\Lambda_n - \Lambda_1}\right)^{1/2} + O(\varepsilon^{3/2}). \tag{3.7}$$

We have now established the result announced in the introduction.

PROPOSITION 3.2. *If $a_1 \in L^\infty(\Omega)$, then there exists a $\delta > 0$ such that $\varepsilon \mapsto \mu(\varepsilon)$ is strictly increasing when $0 \le \varepsilon \le \delta$.*

Proof. If $(a_1\phi_1, \phi_1) < 0$, the result is a consequence of Proposition 3.1, while, if $(a_1\phi_1, \phi_1) > 0$, the result stems from (3.5). Finally, if $(a_1\phi_1, \phi_1) = 0$, we may rely on (3.7). □

In the absence of a uniform lower bound for $\delta(a_1)$ our proposition is just the first step on the way to a fully local result. Such a result would require a careful balance between the radius of convergence, the leading term in the series development of $\lambda_{\pm 1}$, and the magnitude of the remainder. It is the second term that makes this balance tricky for, while the radius of convergence and the size of the remainder depend solely on the size of a_1, the lowest-order terms in $\lambda_{\pm 1}(\varepsilon)$ are highly sensitive to the direction of a_1. If one is willing to limit the direction of the perturbation one may, of course, force the necessary balance. As an example, we note that Proposition 3.1 works in our favor when a_1 lies in the same direction as $-\phi_1^2$. To be precise, recall that $\theta(a_1)$, the angle between a_1 and $-\phi_1^2$, satisfies

$$\cos\theta(a_1) = \frac{(a_1, -\phi_1^2)}{\|a_1\|_2 \|\phi_1^2\|_2}.$$

Now, in the one-dimensional case, given $\delta_1 > 0$ there exists a $\delta_2 > 0$ such that $\sqrt{\Lambda_1}$ minimizes μ on the truncated cone

$$\sqrt{\Lambda_1} + \{a \in L^2(0,\ell) : \cos\theta(a) \ge \delta_1,\ \|a\|_2 \le \delta_2\}.$$

This δ_2 depends on δ_1 and, recalling (3.1), $\|A_0 R(z)\|_{L(X)}$.

4. Comments. We note that similar results may be established for the nonhomogeneous wave equation

$$\rho u_{tt} - \nabla\cdot\xi\nabla u + qu - 2au_t = 0, \quad u(\cdot,t) \in H_0^1(\Omega),$$

as well as those finite-dimensional systems

$$y''(t) + Ky(t) + Dy'(t) = 0, \quad y(t) \in \mathbf{R}^d, \tag{4.1}$$

where D is diagonal and K is symmetric positive definite with a simple least eigenvalue. We note that Langer, Najman, and Vesilić [5] have considered the eigenvalue perturbation problem associated with (4.1) when D is of the form $(1+\varepsilon)C$ for arbitrary, nonhermitian, C and K.

Finally, we explain how the present work relates to the variational properties of the spectral abscissa given by Burke and Overton [2]. In that work, the spectral abscissa μ is viewed as a map from matrix space to $\mathbf{R}$ and a directional derivative for μ is defined and evaluated. There is a close analogy between Theorem 6 of [2] and the main result presented here. The former gives a necessary condition for the directional derivative of μ to be finite and, in the event that the necessary condition holds, gives a lower bound on the directional derivative which, generically, holds with equality. The necessary condition for finiteness corresponds to the condition $(a_1\phi_1, \phi_1) \leq 0$ in the present context. The lower bound which is relevant when the necessary condition holds corresponds to the formula $\frac{1}{2}\operatorname{tr} A_1 P = -(a_1\phi_1, \phi_1)$, given here in Proposition 3.1. The analysis in [2] does not give higher-order terms such as the $O(\varepsilon)$ term in (3.7).

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