

\mathcal{H}^∞ Strong Stabilization via HIFOO, a Package for Fixed-Order Controller Design

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Abstract—We report on our experience with strong stabilization using HIFOO, a toolbox for \mathcal{H}^∞ fixed-order controller design. We applied HIFOO to 21 fixed-order stable \mathcal{H}^∞ controller design problems in the literature, comparing the results with those published for other methods. The results show that HIFOO often achieves good \mathcal{H}^∞ performance with low-order stable controllers, unlike other methods in the literature.

I. INTRODUCTION

By \mathcal{H}^∞ strong stabilization we mean the following: given a linear time-invariant (LTI) multi-input-multi-output (MIMO) system, we are interested in designing a controller that stabilizes the system in closed-loop, reducing the \mathcal{H}^∞ norm of its closed loop transfer function as much as possible, with the additional constraint that the controller is stable. In addition, we require the controller to have a fixed order, specified by the designer. Stable controllers offer several advantages, specifically with respect to disturbance rejection, tracking and modeling uncertainties [39]. They offer protection against sensor failures and actuator saturation [35]. Furthermore, low-order controllers are simpler and therefore easier to implement than full-order controllers, whose order equals the order of the plant, and may therefore offer more confidence for practical use.

Optimal and parameterized suboptimal full-order \mathcal{H}^∞ controllers for LTI MIMO systems can be designed by well-known methods in the literature [14], [17]. However, the practical value of controllers obtained by these methods is limited by the fact that they are full-order and are not generally stable.

There are various methods in the control literature to design stable \mathcal{H}^∞ controllers [6], [12], [10], [22], [29], [37], [38]. All the controllers obtained by these methods have order greater than or equal to the plant order, specifically

- the plant order [10], [29];
- the plant order plus a free parameter Q [7];
- double the plant order [37], [38], [22];
- double the plant order plus the order of a weighting function [6];
- three times the plant order [12].

Thus, these controllers are not practical for high-order plants.

It seems that the reason there is not much literature on designing low-order stable \mathcal{H}^∞ controllers is that the

order of stable controllers could be very large when the plant pole-zero locations in the right-half-plane are close to violating the parity interlacing property. According to [11], it is for this reason that *instead of pursuing minimal-order stable controllers, researchers focus on providing alternative methods to solve the problems*, usually resulting in high-order controllers.

In this paper, we report on our experience applying the HIFOO toolbox (Version 1.75) to fixed-order strong stabilization \mathcal{H}^∞ controller design problems, attempting to minimize the \mathcal{H}^∞ norm of the transfer function for the closed loop plant using a stable controller. This is a difficult optimization problem due to the nonconvexity and nonsmoothness of the objective function and the stability constraint. HIFOO 1.75 uses a hybrid algorithm for nonsmooth, nonconvex optimization based on several techniques to attempt to find fixed-order stable controllers achieving minimal closed-loop \mathcal{H}^∞ norm. HIFOO 1.0 was originally presented in [3], but the original version did not support strong stabilization. HIFOO does not have any restrictions on plant or controller such as nullity or full-rank conditions. It allows the controller order to be specified by the user, unlike other methods in the literature.

HIFOO is freely available MATLAB code¹ and has been designed to be easy to use. It is built on the HANSO optimization package, freely available at the same website. It does not require any external software beyond the MATLAB Control System Toolbox, but it runs much faster if the `linorm` function of the SLICOT package is installed and in the MATLAB path (available commercially from www.slicot.de, but freely available from the HIFOO webpage for noncommercial use with HIFOO using MATLAB running under Windows). HIFOO also makes use of the `quadprog` quadratic programming solver from MOSEK or the MATLAB Optimization Toolbox if it is installed and in the MATLAB path, but this is not required. Our experiments used MATLAB 2006a with `linorm` and `quadprog` installed.

We applied HIFOO to various benchmark plants in the literature and compared our results with published results based on other techniques. Our experience is that HIFOO gives very good experimental results for large sets of data. In particular, we find that it is often possible to obtain stable \mathcal{H}^∞ controllers achieving small closed-loop \mathcal{H}^∞ norm even when the order of the controller is fixed to be much less than the order of the plant.

The rest of the paper is organized as follows. The prob-

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lem of fixed-order strongly stable \mathcal{H}^∞ controller design is described and the optimization method used by HIFOO is summarized in Section II. The benchmark plants are specified in Section III. Our computational results and comparisons with those published for other methods are given in Section IV. Concluding remarks are in Section V.

II. PROBLEM FORMULATION AND OPTIMIZATION METHOD

The state-space equations of a generalized plant G are

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t), \\ z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t), \\ y(t) &= C_2x(t) + D_{21}w(t) + D_{22}u(t)\end{aligned}\quad (\text{II.1})$$

and the state-space realization for the controller K is

$$\begin{aligned}\dot{x}_K(t) &= A_Kx_K(t) + B_Ky(t), \\ u(t) &= C_Kx_K(t) + D_Ky(t),\end{aligned}\quad (\text{II.2})$$

where $A \in \mathcal{R}^{n \times n}$, $D_{12} \in \mathcal{R}^{p_1 \times m_2}$, $D_{21} \in \mathcal{R}^{p_2 \times m_1}$, with other matrices having compatible dimensions, and $A_K \in \mathcal{R}^{n_K \times n_K}$, with B_K, C_K, D_K having dimensions that are compatible with A_K and the generalized plant matrices. The controller order n_K is fixed, so it can be specified by the designer.

The signals (z, w, y, u) respectively represent the regulated outputs, the exogenous inputs (including disturbance and commands), the measured (or sensor) inputs, and the control inputs. The transfer function from the input w to output z is denoted T_{zw} ; see [14] for details. The optimal \mathcal{H}^∞ controller design can be formulated as minimization of the closed loop \mathcal{H}^∞ norm function

$$\inf_{K \text{ stabilizing}} \|T_{zw}\|_\infty, \quad (\text{II.3})$$

where the constraint specifies that K internally stabilizes the closed-loop system.

In this paper, we impose the additional constraint that the controller is stable, so that we wish to minimize

$$\inf_{K \text{ stabilizing and } K \text{ stable}} \|T_{zw}\|_\infty. \quad (\text{II.4})$$

Let us use $\alpha(X)$ to denote the spectral abscissa of a matrix X , i.e., the largest of the real parts of the eigenvalues. Thus, not only do we require that $\alpha(A_{CL}) < 0$, where A_{CL} is the closed-loop system matrix, but we also require that $\alpha(A_K) < 0$. The feasible set for A_K , that is the set of stable matrices, is not a convex set and has a boundary that is not smooth. It has been studied extensively, see e.g. [5], [23].

As with previous versions [3], [31] HIFOO uses two phases: stabilization and performance optimization. In the stabilization phase, HIFOO 1.75 proceeds to minimize $\max(\alpha(A_{CL}), \epsilon\alpha(A_K))$, where ϵ is a positive parameter that will be described shortly, until it finds a controller K for which this quantity is negative, i.e., the controller is stable and stabilizes the closed loop system. If it cannot find such a controller, HIFOO will return with a message to that effect.

In the performance optimization phase, HIFOO 1.75 looks for a local minimizer of

$$f(K) = \begin{cases} \infty & \text{if } \max(\alpha(A_{CL}), \alpha(A_K)) \geq 0 \\ \max(\|T_{zw}\|_\infty, \epsilon\|K\|_\infty) & \text{otherwise,} \end{cases} \quad (\text{II.5})$$

where

$$\|K\|_\infty = \sup_{\Re s=0} \|C_K(sI - A_K)^{-1}B_K + D_K\|_2. \quad (\text{II.6})$$

The motivation for the introduction of ϵ is that the principal design goal is to stabilize the closed loop system and minimize $\|T_{zw}\|_\infty$, indicating that ϵ should be relatively small, but the $\epsilon\|K\|_\infty$ term prevents the \mathcal{H}^∞ norm of the controller from growing too large, which the stability constraint by itself will not. Because of the stabilization phase, the performance optimization phase begins with a finite value for $f(K)$. When it subsequently encounters an instance of K for which $f(K) = \infty$ it is rejected by the line search which insists on a reduction in the objective at every iteration.

The optimization code called by HIFOO in both phases is HANSO, which implements a hybrid algorithm for non-smooth, nonconvex optimization, based on the following elements: a quasi-Newton algorithm (BFGS) provides a fast way to approximate a local minimizer; a local bundle method attempts to verify local optimality for the best point found by BFGS, and if this does not succeed, gradient sampling [4], [2] attempts to refine the approximation of the local minimizer, returning a rough optimality measure. The local bundle and gradient sampling methods are not invoked if the quadratic programming code `quadprog` is not in the MATLAB path. All three of these optimization techniques use gradients which are automatically computed by HIFOO. No effort is made to identify the exceptional points where the gradients fail to exist. The algorithms are not defeated by the discontinuities in the gradients at exceptional points. The BFGS phase builds a highly ill-conditioned Hessian approximation matrix, and the bundle and gradient sampling final phases search for a point in parameter space for which a convex combination of gradients at nearby points has small norm. More details are given in [3].

Because HIFOO uses randomized starting points, and also the gradient sampling phase involves randomization, the same results are not obtained every time HIFOO is run. In the results reported below, we made multiple runs setting ϵ to 10^{-2} , 10^{-3} , 10^{-4} , 10^{-5} , and 10^{-6} , and running each case 10 times. Each result in the tables in Section IV reports the lowest value for $\|T_{zw}\|_\infty$ obtained over all these runs. We did not attempt to compare the running times of different methods. In our view, one of the biggest advantages of HIFOO is its ease of use. Generally, the running time requirements for computation of controllers are not nearly as important as the performance and safety aspects of the computed controllers. Implementing any controller is far more work than computing it, so the key aspect of running time in computing a controller is that it should not be longer than the designer is willing to wait. For this reason HIFOO accepts an option, `options.cpumax`, which controls the

running time. Better performance may be obtained if a larger value of `options.cpumax` is specified. We set `options.cpumax` to 300 (5 minutes) in all of our tests.

III. BENCHMARK PROBLEMS

Benchmark examples for stable \mathcal{H}^∞ controller design were chosen from both applied and academic test problems, as follows.

- 1) **Zeren-Özbay Example:** A 5th-order plant given in [38]. For this example, the optimal full-order \mathcal{H}^∞ controller is unstable and furthermore the central controller [14] for any closed-loop \mathcal{H}^∞ norm is unstable.
- 2) **Cao-Lam Example:** A 2nd-order plant given in [8].
- 3) **Choi-Chung Example:** A 4th-order plant given in [10].
- 4) **Four-Disk System:** An 8th-order four-disk system with noncolocated sensors and actuators given in [1], [6].
- 5) **AC8:** A 9th-order state-space model of the linearized vertical plane dynamics of an aircraft [18];
- 6) **HE1:** A 4th-order model of the longitudinal motion of a VTOL helicopter [27];
- 7) **REA2:** A 4th-order chemical reactor model [25];
- 8) **AC10:** A 55th-order aeroelastic model of a modified Boeing B-767 airplane [13];
- 9) **BDT2:** An 82nd-order realistic model of a binary distillation tower [33];
- 10) **HF1:** A 130th-order one-dimensional model for heat flow in a thin rod [24];
- 11) **CM4:** A 240th-order cable mass model for nonlinear dynamic response of a relief valve protecting a pneumatic system from overpressure [32];
- 12) **PA:** A 5th-order model of a piezoelectric bimorph actuator system [9];
- 13) **HIMAT:** A 20th-order model of an experimental highly maneuverable (HIMAT) airplane [20];
- 14) **VSC:** A 4th-order quarter-car model representing characteristics of a real suspension system [30];
- 15) **AUV:** 3rd, 5th and 6th-order linearized models of an autonomous underwater vehicle for speed, heading and depth autopilots respectively [16];
- 16) **Enns' Example:** An 8th-order plant used as an academic test problem for designing reduced-order \mathcal{H}^∞ controllers [15];
- 17) **Wang's Example:** A 4th-order plant used as a theoretical benchmark problem for designing reduced-order \mathcal{H}^∞ controllers [36], Example 6.2.

Examples 1 – 4 are collected from various papers specifically concerned with strongly stable \mathcal{H}^∞ controller design. The plants 5 – 17 were collected in [21] as benchmark examples for fixed-order \mathcal{H}^∞ controller design without any stability constraint on the controller. Examples 5 – 15 are taken from real applications and 16 – 17 are academic test problems. The problem data for examples 5–12 are obtained from the `COMPLeIB` library [28] and those for examples 13 – 17 are collected from various papers in the literature. In

the runs reported in the next section, the strong stabilization constraint is imposed for all examples. We do not give running times in this paper, but times for the results reported in [21] are available on the web.²

IV. RESULTS ON BENCHMARK PROBLEMS

In Tables I-VI, we compare the performance of HIFOO with other methods from the strong stabilization literature on examples 1-4. Tables VII-VIII show results obtained by HIFOO when the strong stabilization constraint is imposed on examples 5-17 (there are no results from the literature to compare for these examples). In all the tables, the controller order is shown by n_K , and γ_{n_K} shows the \mathcal{H}^∞ performance achieved for this order using the method indicated. In Tables I-VI, the lines in the table shaded in *light gray* show results for the various strong stabilization methods in the literature, which all produce controllers with order greater than the order of the plant, as mentioned in Section I. In all the tables, the line shaded in *dark gray* (labeled **full** in Tables I-VI), shows, for n_K , the order of the plant and, for γ_{n_K} , the \mathcal{H}^∞ performance for the optimal full-order controller computed using the `hinfsyn` routine of [17] (see also [14]). The \mathcal{H}^∞ performance of the full-order controller is a lower bound for the achievable \mathcal{H}^∞ norm by any order controller and is therefore a measure of performance for all methods. The *unshaded* lines below the full-order controller line show the results obtained by HIFOO for various specified controller orders. The last column in Tables I-VI indicates whether the controller is stable.

A. Zeren-Özbay Example

TABLE I
COMPARISON ON ZEREN-ÖZBAY EXAMPLE

n_K	γ_{n_K}	Methods	Controller Stability
10	42.51	[38]	Stable
10	35.29	[22]	Stable
6	34.44	[7]	Stable
5	34.24	full	Unstable
5	34.81	HIFOO	Stable
4	34.97	HIFOO	Stable
3	34.94	HIFOO	Stable
2	41.16	HIFOO	Stable
1	57.32	HIFOO	Stable

Results for this example are given in Table I. The plant order is 5 and the optimal full-order controller is unstable. The performance of the method [7] is good as it finds a stable 6th order controller with a closed-loop \mathcal{H}^∞ norm close to the optimal full-order performance. However, HIFOO finds a stable 3rd order controller with nearly the same \mathcal{H}^∞ norm.

B. Cao-Lam Example

Results for this example are given in Table II. The plant order is 2 and, as in the previous example, the optimal full-order \mathcal{H}^∞ controller is unstable. The method of [11] finds a

²<http://www.cs.nyu.edu/overton/papers/pdf/acc08times.pdf>

TABLE II
COMPARISON ON CAO-LAM EXAMPLE

n_K	γ_{n_K}	Methods	Controller Stability
8	1.29338	[11]	Stable
4	1.36994	[29]	Stable
4	1.36957	[22]	Stable
4	1.36814	[11]	Stable
2	1.29022	full	Unstable
2	1.36957	HIFOO	Stable
1	1.36957	HIFOO	Stable

stable controller with nearly the same \mathcal{H}^∞ performance, but it uses an 8th order controller. HIFOO finds a stable 1st order controller with less than 10% increase in \mathcal{H}^∞ performance, approximately the same as that found by the other methods for 4th order controllers.

C. Choi-Chung Example

TABLE III
COMPARISON ON CHOI-CHUNG EXAMPLE

n_K	γ_{n_K}	Methods	Controller Stability
16	25.430	[11]	Stable
12	21.787	[12]	Stable
8	43.167	[10]	Stable
8	37.551	[37]	Stable
8	32.557	[22]	Stable
8	24.790	[11]	Stable
4	12.015	full	Unstable
4	16.612	HIFOO	Stable
3	16.486	HIFOO	Stable
2	20.797	HIFOO	Stable
1	62.638	HIFOO	Stable

Results for this example are given in Table III. We see again that the stable \mathcal{H}^∞ controller design methods in the literature are conservative in terms of controller order. HIFOO achieves better \mathcal{H}^∞ performance than the other methods with a lower-order stable controller.

D. Four-Disk System

The results for the Four-Disk System with three different parameter values are shown in Tables IV-VI. One can see that the design objectives (stability and low closed-loop \mathcal{H}^∞ norm) are achieved by HIFOO using low-order controllers. HIFOO achieves the same \mathcal{H}^∞ performance as the other methods with a 5th order stable controller whereas the controllers obtained by the other methods have order 16, 16 and 24 respectively.

E. Other Benchmark Examples

Results obtained using HIFOO to find stable controllers for examples 5 – 17 are shown in Table VII and VIII. The examples are grouped according to plant order: Table VII shows low to medium-order plants and Table VIII shows

TABLE IV
COMPARISON ON FOUR-DISK SYSTEM, $\beta = 10^{-1}$

n_K	γ_{n_K}	Methods	Controller Stability
24	0.237	[6]	Stable
16	0.245	[37]	Stable
16	0.241	[22]	Stable
8	0.232	full	Unstable
8	0.235	HIFOO	Stable
7	0.236	HIFOO	Stable
6	0.236	HIFOO	Stable
5	0.235	HIFOO	Stable
4	0.274	HIFOO	Stable
3	0.307	HIFOO	Stable
2	0.347	HIFOO	Stable
1	0.649	HIFOO	Stable

TABLE V
COMPARISON ON FOUR-DISK SYSTEM, $\beta = 10^{-2}$

n_K	γ_{n_K}	Methods	Controller Stability
24	0.151	[6]	Stable
16	0.178	[37]	Stable
16	0.176	[22]	Stable
8	0.141	full	Unstable
8	0.152	HIFOO	Stable
7	0.153	HIFOO	Stable
6	0.153	HIFOO	Stable
5	0.152	HIFOO	Stable
4	0.212	HIFOO	Stable
3	0.276	HIFOO	Stable
2	0.314	HIFOO	Stable
1	0.634	HIFOO	Stable

higher-order plants. For the low to medium-order plants, we report results for strong stabilization with order ranging from 1 to the order of the plant. For the higher-order plants, we restricted the order of the controller to 8.

The performance of HIFOO is very good for low and medium-order plants. In most cases, the optimal closed-loop full-order \mathcal{H}^∞ performance is achieved or nearly achieved by 1st-3rd order stable controllers, even though the full-order controller is not necessarily stable. In general, HIFOO shows that it is often possible to find a stable low-order controller without greatly sacrificing closed-loop \mathcal{H}^∞ performance.

HIFOO also performs successfully on higher-order plants as shown in Table VIII. These examples are numerically difficult and it is sometimes difficult to calculate the optimal full-order \mathcal{H}^∞ performance with well-known robust algorithms. The plant AC10 is particularly difficult to stabilize so we needed more runs than used for the other plants, building higher order controllers with lower order ones as initial search points; we omit the details. These results clearly demonstrate that HIFOO is very effective not only for simple plants but real-life high-order plants that arise in industrial applications.

TABLE VI
COMPARISON ON FOUR-DISK SYSTEM, $\beta = 10^{-3}$

n_K	γ_{n_K}	Methods	Controller Stability
24	0.132	[6]	Stable
16	0.170	[37]	Stable
16	0.170	[22]	Stable
8	0.122	full	Unstable
8	0.142	HIFOO	Stable
7	0.143	HIFOO	Stable
6	0.145	HIFOO	Stable
5	0.154	HIFOO	Stable
4	0.208	HIFOO	Stable
3	0.274	HIFOO	Stable
2	0.314	HIFOO	Stable
1	0.634	HIFOO	Stable

V. CONCLUDING REMARKS

We applied the HIFOO Toolbox to 21 strongly-stable design problems, taken from a mix of industrial and academic test problems. The performance of HIFOO is better than existing results in the literature in most cases, even when the specified controller order is low. We conclude that HIFOO is an effective method for fixed-order stable \mathcal{H}^∞ controller design, giving flexibility to the designer to specify the controller order and generally obtaining good performance. HIFOO, which is written in MATLAB, is easy to use and is freely available on the web.

VI. ACKNOWLEDGMENTS

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REFERENCES

- [1] D. S. Bernstein and W. M. Haddad, "LQG control with an H_∞ performance bound: A riccati equation approach," *IEEE Transactions on Automatic Control*, vol.34, pp. 293–305, 1989.
- [2] J. V. Burke, D. Henrion, A. S. Lewis and M. L. Overton, "Stabilization via nonsmooth, nonconvex optimization," *IEEE Transactions on Automatic Control*, vol.51, pp. 1760–1769, 2006.
- [3] J. V. Burke, D. Henrion, A. S. Lewis, M. L. Overton, "HIFOO - A MATLAB package for fixed-order controller design and H_∞ optimization", *IFAC Symposium on Robust Control Design*, Toulouse, France, July 2006.
- [4] J. V. Burke, A. S. Lewis and M. L. Overton, "A robust gradient sampling algorithm for nonsmooth nonconvex optimization," *SIAM Journal on Optimization*, vol.15, pp. 751–779, 2005.
- [5] J. V. Burke and M. L. Overton, "Variational Analysis of Non-Lipschitz Spectral Functions," *Mathematical Programming*, vol.90, pp. 317–352, 2001.
- [6] D. U. Campos-Delgado and K. Zhou, " \mathcal{H}^∞ Strong stabilization," *IEEE Transactions on Automatic Control*, vol.46, pp.1968–1972, 2001.
- [7] D. U. Campos-Delgado and K. Zhou, "A parametric optimization approach to \mathcal{H}^∞ and \mathcal{H}^2 strong stabilization," *Automatica*, vol.39, No.7, pp. 1205–1211, 2003.
- [8] Y. Y. Cao and J. Lam, "On simultaneous H_∞ control and strong H_∞ stabilization," *Automatica*, vol.36, No.6, pp. 859–865, 2000.
- [9] B. M. Chen, *H_∞ Control and Its Applications*, vol. 235 of Lectures Notes in Control and Information Sciences, Springer Verlag, New York, Heidelberg, Berlin, 1998.

TABLE VII
RESULTS OF HIFOO ON OTHER BENCHMARK EXAMPLES (LOW-MEDIUM ORDER PLANTS). THE FIRST LINE SHOWS THE PLANT ORDER AND FULL-ORDER CONTROLLER PERFORMANCE, WITH AN ASTERISK INDICATING AN UNSTABLE FULL-ORDER CONTROLLER.

HE1		REA2		VSC		Wang's Ex	
n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}
4*	0.0737	4*	1.134	4*	3.216	4	50.640
4	0.0924	4	1.134	4	3.421	4	50.640
3	0.0924	3	1.134	3	3.408	3	50.640
2	0.0925	2	1.134	2	3.437	2	50.642
1	0.1235	1	1.134	1	3.362	1	50.645

AUV Speed		AUV Heading		AUV Depth		PA	
n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}
3	0.954	5	0.954	6	0.955	5	$1.0 \cdot 10^{-6}$
3	0.955	5	0.954	6	0.955	5	$4.1 \cdot 10^{-4}$
2	0.955	4	0.954	5	0.955	4	$3.6 \cdot 10^{-4}$
1	0.955	3	0.954	4	0.955	3	$3.6 \cdot 10^{-4}$
		2	0.954	3	0.955	2	$3.5 \cdot 10^{-4}$
		1	0.954	2	0.955	1	$3.5 \cdot 10^{-4}$
				1	0.957		

Enns' Ex		AC8	
n_K	γ_{n_K}	n_K	γ_{n_K}
8	1.127	9	1.617
8	1.130	9	1.617
7	1.131	8	1.617
6	1.130	7	1.617
5	1.131	6	1.617
4	1.167	5	1.617
3	1.200	4	1.617
2	1.244	3	1.617
1	1.426	2	1.626
		1	1.651

- [10] Y. Choi and W.K. Chung, "On the Stable \mathcal{H}^∞ Controller Parameterization Under Sufficient Condition," *IEEE Transactions on Automatic Control*, vol.46, pp. 1618–1623, 2001.
- [11] Y. S. Choi, J. L. Leu and Y. C. Chung, "Stable controller design for MIMO systems: an LMI approach," *IET Control Theory Applications*, vol.1, pp. 817–829, 2007.
- [12] Y. S. Chou, T. Z. Wu and J. L. Leu, "On Strong Stabilization and \mathcal{H}^∞ Strong-Stabilization Problems," *Proc. Conference on Decision and Control*, pp. 5155–5160, 2003.
- [13] E. J. Davison, "Benchmark problems for control system design," technical report, International Federation of Automatic Control, 1990. Report of the IFAC Theory Committee.
- [14] J. Doyle, K. Glover, P. Khargonekar, and B. A. Francis, "State-space solutions to standard \mathcal{H}_2 and \mathcal{H}_∞ control problems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, 1989.
- [15] D.F. Enns, "Model Reduction for Control System Design," Ph.D. Dissertation, Stanford University, 1984.
- [16] Z. Feng and R. Allen, "Reduced order H_∞ control of an autonomous underwater vehicle," *Control Engineering Practice*, vol.12, pp. 1511–1520, 2004.
- [17] G. J. Galas, A. K. Packard, M. G. Safonov and R. Y. Chiang, *Robust Control Toolbox 3.3*, The Mathworks, Natick, 2008.
- [18] D. Gangsaas, K. Bruce, J. Blight, and U.-L. Ly, "Application of modern synthesis to aircraft control: Three case studies," *IEEE Transactions on Automatic Control*, vol.31, pp. 995–1014, 1986.
- [19] H. Gao, J. Lam and C. Wang, "Controller reduction with H_∞ error performance: continuous- and discrete-time cases", *International Journal of Control*, vol.79, pp. 604–616, 2006.
- [20] P. J. Goddard and K. Glover, "Controller Approximation: Approaches for preserving H_∞ Performance," *IEEE Transactions on Automatic Control*, vol.36, pp. 858–871, 1998.

TABLE VIII

RESULTS OF HIFOO ON OTHER BENCHMARK EXAMPLES (HIGH-ORDER PLANTS). THE FIRST LINE SHOWS THE PLANT ORDER AND FULL-ORDER CONTROLLER PERFORMANCE, WITH AN ASTERISK INDICATING AN UNSTABLE FULL-ORDER CONTROLLER.

HIMAT		AC10		BDT2		HF1	
n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}	n_K	γ_{n_K}
20	0.970	55*	0.633	82*	0.234	130	0.447
8	1.060	8	8.984	8	0.531	8	0.447
7	1.068	7	9.003	7	0.542	7	0.447
6	1.061	6	9.376	6	0.534	6	0.447
5	1.067	5	9.570	5	0.559	5	0.447
4	1.072	4	9.869	4	0.604	4	0.447
3	1.109	3	9.869	3	0.578	3	0.447
2	2.155	2	9.869	2	0.576	2	0.447
1	2.782	1	10.863	1	0.643	1	0.447

CM4	
n_K	γ_{n_K}
240	0.816
8	0.824
7	0.819
6	0.818
5	0.817
4	0.818
3	0.817
2	0.817
1	0.817

- [21] S. Gumussoy and M. L. Overton, "Fixed-Order H-infinity Controller Design via HIFOO, a Specialized Nonsmooth Optimization Package," *Proc. of American Control Conference*, pp. 2750–2754, 2008.
- [22] S. Gumussoy and H. Özbay, "Remarks on strong stabilization and stable \mathcal{H}^∞ controller design," *IEEE Transactions on Automatic Control*, vol.50, pp. 2083–2087, 2005.
- [23] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory I: Modelling, State Space Analysis, Stability and Robustness*, Springer, 2005.
- [24] A. S. Hodel, K. Poolla, and B. Tension, "Numerical solution of the Lyapunov equation by approximate power iteration," *Linear Algebra Applications*, vol.236, pp. 205–230, 1996.
- [25] Y. S. Hung and A. G. J. MacFarlane, *Multivariable feedback: A classical approach*, Lectures Notes in Control and Information Sciences, Springer Verlag, New York, Heidelberg, Berlin, 1982.
- [26] D. Kavranoglu and S.H. Al-Amer, "New efficient frequency domain algorithm for \mathcal{H}^∞ approximation with applications to controller reduction," *IEE Proceedings-Control Theory Applications*, vol.148, pp. 383–390, 2001.
- [27] L. H. Keel, S. P. Bhattacharyya, and J. W. Howze, "Robust control with structured perturbations," *IEEE Transactions on Automatic Control*, vol.36, pp. 68–77, 1988.
- [28] F. Leibfritz, "COMPL_eIB, constraint matrix-optimization problem library A collection of test examples for nonlinear semidefinite programs, control system design and related problems," Universität Trier, Tech. Rep., 2003. www.compleib.de
- [29] P.H. Lee and Y.C. Soh, "Synthesis of stable \mathcal{H}^∞ controller via the chain scattering framework," *System and Control Letters*, vol.46, pp.1968–1972, 2002.
- [30] J. S. Lin, and I. Kanellakopoulos, "Nonlinear design of active suspensions," *IEEE Control Systems Magazine*, vol.17, pp. 45–59, 1997.
- [31] M. Millstone, "HIFOO 1.5: Structured control of linear systems with a non-trivial feedthrough", M.S. thesis, Courant Institute of Mathematical Sciences, New York University, 2006.
- [32] A. H. Nayfeh, J. F. Nayfeh, and D. T. Mook, "On methods for continuous systems with quadratic and cubic nonlinearities," *Nonlinear Dynamics*, vol.3, pp. 145–162, 1992.
- [33] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control*, John Wiley & Sons, 1996.
- [34] M. C. Smith and K. P. Sondergeld, "On the order of stable compensators," *Automatica*, vol.22, pp. 127–129, 1986.
- [35] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.
- [36] J.-Z. Wang and L. Huang, "Controller order reduction with guaranteed performance via coprime factorization", *International Journal of Robust and Nonlinear Control*, vol.13, pp. 501–517, 2003.
- [37] M. Zeren and H. Özbay, "On the synthesis of stable \mathcal{H}^∞ controllers," *IEEE Transactions on Automatic Control*, vol.44, pp. 431–435, 1999.
- [38] M. Zeren and H. Özbay, "On the strong stabilization and stable H^∞ -controller design problems for MIMO systems," *Automatica*, vol.36, pp. 1675–1684, 2000.
- [39] K. Zhou, J.C. Doyle and K. Glover, *Robust and Optimal Control*, Upper Saddle River: Prentice-Hall, 1996.