Social*Networks

QUIZ #7

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Q1. [10] The (Little) Monty Hall Problem: You are a contestant on a game show, and you have to choose one of three doors. Behind one door is a car, behind the others, goats. It is assumed that car is preferable to goats. You pick a door (say, number 2), and the host, who knows what is behind each door, opens one of the other two doors (say, number 3), and reveals a goat.

He then says to you, "Do you want to change your mind, and pick door number 1?" What should be your response.

SOLN.1 *You should change your mind. Here is a rigorous argument, using Bayes' Theorem.*

Consider the discrete random variables, all taking values in the set of door numbers $\{1, 2, 3\}$*:*

$$\begin{aligned} C &: \text{ the number of the door hiding the Car,} \\ S &: \text{ the number of the door Selected by the player, and} \\ H &: \text{ the number of the door opened by the Host.} \end{aligned}$$

As the host's placement of the car is random, all values $c \in \{1, 2, 3\}$ *of* C *are equally likely. The initial (unconditional) probability distribution of* C *is then* $Pr[C = c] = \frac{1}{3}$*, for every value of* c*. Further, as the initial choice of the player is independent of the placement of the car, variables* C *and* S *are independent. Hence the conditional probability of* $C = c$ *given* $S = s$ *is* $Pr[C = c|S = s] = Pr[C = c]$*, for every value of* c *and* s*. The host'ss behavior is reflected by the values of the conditional probability of* $H = h$ *given* $C = c$ *and* $S = s$*:*

$$Pr[H = h|C = c \wedge S = s]$$
$$= \begin{cases} 0 & \text{if } h = s \text{ (host cannot open the door selected);} \\ 0 & \text{if } h = c \text{ (host cannot open the door with car);} \\ \frac{1}{2} & \text{if } s = c \wedge h \neq s \text{ (host can open any of the two remaining doors);} \\ 1 & \text{if } s \neq c \wedge h \neq s \wedge h \neq c \text{ (host can open the only remaining door).} \end{cases}$$

The player can then use Bayes' theorem to compute the probability of finding the car behind a door, once the initial selection has been made and *the host has opened one door. This is the conditional probability of $C = c$ given $H = h$ and $S = s$:*

$$Pr[C = c|H = h \wedge S = s] = \frac{Pr[H = h|C = c \wedge S = s]Pr[C = c|S = s]}{Pr[H = h|S = s],}$$

where the denominator is computed using the law of total probability as the marginal probability

$$Pr[H = h|S = s] = \sum_c Pr[H = h|C = c \wedge S = s]Pr[C = c|S = s].$$

Thus, if the player initially selects door 2, and the host opens door 1, the probability of winning by not switching is

$$Pr[C = 2|H = 3 \wedge S = 2] = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{1}{3};$$

and the probability of winning by switching is

$$Pr[C = 1|H = 3 \wedge S = 2] = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{2}{3}.$$

Q2. [10] The Big Monty Hall Problem: Same problem, but now instead of three, there are hundred doors; there is just one car, and ninety-nine goats. You pick a door (say, number 2), and the host, who knows what is behind each door, opens ninety-eight of the other ninety-nine other doors (say, numbers 3 through 100), and reveals goats.

He then says to you, "Do you want to change your mind, and pick door number 1?" What should be your response.

SOLN.2 *Initially, your chance of choosing the car was one in hundred, and if you don't change your mind, then the probability of you having already picked the correct door remains unchanged. Thus, the probability that if you change to the other unopened door you will find a car is ninety-nine in hundred. Hence you should change.*

More tediously, you can argue as follows. Assume (without loss of generality) that the car is behind door 2; suppose you had picked 2, then in this case changing your mind is a bad idea – i.e., loss of a car. But if you had picked any door other than 2 (say 1), then you would win a car by changing your mind. So you should change your mind.