Programming Assignment 4

Assigned: July 11
Due: July 25

Problem 1: Exact probabilistic calculation

Consider the following situation: There is a network of links between nodes — that is, an undirected graph whose vertices are the nodes and whose edges are the links. Each links fails with probability P. You wish to know what is the probability that K pairs of nodes are connected, for K between 0 and $N(N-1)/2$ where N is the number of nodes.

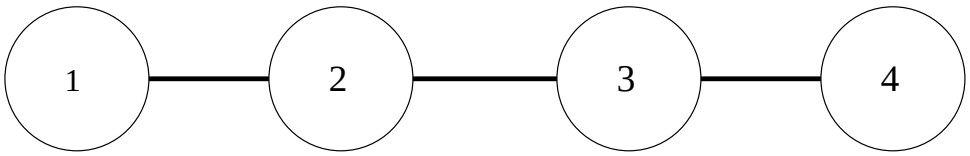
Part A

Write a program `NumConnectedPairs(E,P)` that takes as input a symmetric array `E` of 1's and 0's corresponding to the network and a probability `P` that any given link will fail. It should return a probability distribution `D` such that `D[I]` is the probability that exactly `I` pairs of cities are connected. This omits the probability that no cities are connected, which is always just $\texttt{P}^{\texttt{L}}$ where L is the number of links.

There is no polynomial-time solution to this problem, so you should just use exhaustive enumeration; consider all possible subsets of the links, and for each, compute their probability and the number of pairs of nodes connected, and add up the total probability for each.

For a given subnetwork, the number of pairs connected can be computed as follows: Use a depth-first search to divide the network into connected components. Then each connected component of size K connects $K(K-1)/2$ pairs of nodes.

For instance, consider the simple network of three cities shown below and suppose that each link can fail with probability 0.2.



The corresponding function call would be

```
E=[0,1,0,0; 1,0,1,0; 0,1,0,1; 0,0,1,0];
P=0.2;
NumConnectedPairs(E,P)
```

Since there are three links, there are 8 possible sub-networks. These are shown in the table below.

Case	Active links	Pairs connected	Probability
1.	1-2, 2-3, 3-4	6	0.512
2.	1-2, 2-3	3	0.128
3.	1-2, 3-4	2	0.128
4.	1-2	1	0.032
5.	2-3, 3-4	3	0.128
6.	2-3	1	0.032
7.	3-4	1	0.032
8.	$\emptyset$	0	0.008

Therefore the probability is 0.096 that 1 pair of cities is connected; 0.128 that 2 pairs are connected; 0.256 that 3 pairs are connected; and 0.512 that 6 pairs are connected. So the function should return [0.096, 0.128, 0.256, 0, 0, 0.512]

Part B

In the same situation as part A, write a function `ProbConnected(E,P,PairA)` where `E` and `P` are the same as in part A, and `PairA` is a pair of nodes. For instance, in the above example with the same values of `E` and `P`, `ProbConnected(E,P,[1,2])` should return 0.8 and `ProbConnected(E,P,[1,4])` should return 0.512.

In this case the probability is easily computed, but that will not be the case in general; you should carry out the same enumeration of cases as in part A.

Part C

In the same situation as part A, write a function `CondProbConnected(E,P,PairA,PairB,BConn)` where `E` and `P` are the same as in part A; and `PairA` and `PairB` are each a pair of nodes. If `BConn=true` then the function returns the conditional probability that `PairA` is connected given that `PairB` is connected; if `BConn=false` then the function returns the conditional probability that `PairA` is connected given that `PairB` is *not* connected.

For instance, in the above example

`CondProbConnected(E,P,[1,2],[1,4],1)` should return 1.
`CondProbConnected(E,P,[1,2],[1,4],0)` should return 0.5902 (Cases 2,3,4 out of cases 2-8).
`CondProbConnected(E,P,[1,4],[1,2],1)` should return 0.64.
`CondProbConnected(E,P,[1,4],[1,2],0)` should return 0.
`CondProbConnected(E,P,[2,4],[1,3],1)` should return 0.8.
`CondProbConnected(E,P,[2,4],[1,3],0)` should return 0.3556 (Case 5 out of cases 3-8).

For all three parts of this problem it is OK to write an exponential space solution (i.e. one that essentially generates the table above as a data structure), but it is certainly better to write code that requires only reasonable amounts of memory, though an exponential amount of time.

Problem 2: Monto Carlo probabilistic calculation

Consider the same problem as in problem 1, of a network with links that fail at random.

Part A

Write a function `MCProbConnected(E,P,PairA,N)` with the same functionality as the function `ProbConnected(E,P,PairA)` that you wrote for programming assignment 4, but which uses Monte Carlo search. The parameter `N` is the number of Monte Carlo attempts.

This is quite simple.

```
Count = 0;
for I = 1:N
    generate a random subgraph S of E by going through every edge in E
       and having it fail with probability P;
    if U and V are connected in S, then COUNT++;
end
return COUNT/N;
```

Your code should run in time proportional to NV^2 where V is the number of vertices in the graph. (Actually, it can be done in time $O(N(V+E)+V^2)$ where E is the number of edges; but that would involve converting the adjacency array representation of the graph to an adjacency list representation; and I'm not asking you to do that.) The program should require space proportional to the size of the graph. Points will be deducted if it exceeds either these.

Part B

Write a function `ExpectedNumConnectedPairs(E,P,N)` that uses Monte Carlo search to compute the expected number of connected pairs.

This is much the same as in part A.

```
Total = 0;
for I = 1:N
    generate a random subgraph S of E by going through every edge in E
       and having it fail with probability P;
    Total = Total + the number of pairs connected in S;
end
return Total/N;
```

Part C (Experimental, not more programming)

Consider a graph on 11 vertices that is just a straight line: A is connected to B, B is connected to A and C; C is connected to B and D; ... J is connected to I and K; K is connected to J.

With this graph, how many sample points are required to get an answer for part B accurate to within 1% for (a) P=0.1; (b) P=0.6? (Note that you can get the exact answer by using `NumConnectedPairs` from problem 1.) Include your answer to this part as a comment in the code for Part B.