Problem 1 (CLRS 23.1-6). (1 point) Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

Problem 2 (CLRS 23-1). (3 points) Let $G = (V, E)$ be an undirected, connected graph whose weight function is $w : E \rightarrow \mathbb{R}$, and suppose that $|E| \geq |V|$ and all edge weights are distinct.

We define a second-best minimum spanning tree as follows. Let $T$ be the set of all spanning trees of $G$, and let $T'$ be a minimum spanning tree of $G$. Then a second-best minimum spanning tree is a spanning tree $T$ such that $w(T) = \min_{T'' \in T - \{T'\}} \{w(T'')\}$.

(a) Show that the minimum spanning tree is unique, but that the second-best minimum spanning tree need not be unique.

(b) Let $T$ be the minimum spanning tree of $G$. Prove that $G$ contains edges $(u, v) \in T$ and $(x, y) \notin T$ such that $T - \{(u, v)\} \cup \{(x, y)\}$ is a second-best minimum spanning tree of $G$.

(c) Let $T$ be a spanning tree of $G$ and, for any two vertices $u, v \in V$ let $\max[u, v]$ denote an edge of maximum weight on the unique simple path between $u$ and $v$ in $T$. Describe an $O(V^2)$-time algorithm that, given $T$, computes $\max[u, v]$ for all $u, v \in V$.

(d) Give an efficient algorithm to compute the second-best minimum spanning tree of $G$.

Problem 3 (CLRS 24.3-6). (2 points) We are given a directed graph $G = (V, E)$ on which each edge $(u, v) \in E$ has an associated value $r(u, v)$, which is a real number in the range $0 \leq r(u, v) \leq 1$ that represents the reliability of a communication channel from vertex $u$ to vertex $v$. We interpret $r(u, v)$ as the probability that the channel from $u$ to $v$ will not fail, and we assume that these probabilities are independent. Give an efficient algorithm to find the most reliable path between two given vertices.

Problem 4 (CLRS 24-2). (2 points) A $d$-dimensional box with dimensions $(x_1, x_2, ..., x_d)$ nests within another box with dimensions $(y_1, y_2, ..., y_d)$ if there exists a permutation $\pi$ on $\{1, 2, ..., d\}$ such that $x_{\pi(1)} < y_1, x_{\pi(2)} < y_2, ..., x_{\pi(d)} < y_d$.

(a) Argue that the nesting relation is transitive.

(b) Describe an efficient method to determine whether or not one $d$-dimensional box nests inside another.

(c) Suppose that you are given a set of $n$ $d$-dimensional boxes $\{B_1, B_2, ..., B_n\}$. Give an efficient algorithm to find the longest sequence $\langle B_{i_1}, B_{i_2}, ..., B_{i_k}\rangle$ of boxes such that $B_{i_j}$ nests within $B_{i_{j+1}}$ for $j = 1, 2, ..., k - 1$. Express the running time of your algorithm in terms of $n$ and $d$. 

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Problem 5 (CLRS 24-3). (3 points) Arbitrage is the use of discrepancies in currency exchange rates to transform one unit of a currency into more than one unit of the same currency. For example, suppose that 1 U.S. dollar buys 49 Indian rupees, 1 Indian rupee buys 2 Japanese yen, and 1 Japanese yen buys 0.0107 U.S. dollars. Then, by converting currencies, a trader can start with 1 U.S. dollar and buy $49 \times 2 \times 0.0107 = 1.0486$ U.S. dollars, thus turning a profit of 4.86 percent.

Suppose that we are given $n$ currencies $c_1, c_2, ..., c_n$ and an $n \times n$ table $R$ of exchange rates, such that one unit of currency $c_i$ buys $R[i, j]$ units of currency $c_j$.

(a) Give an efficient algorithm to determine whether or not there exists a sequence of currencies $\langle c_{i_1}, c_{i_2}, ..., c_{i_k} \rangle$ such that

$$R[i_1, i_2] \cdot R[i_2, i_3] \cdot ... \cdot R[i_{k-1}, i_k] \cdot R[i_k, i_1] > 1.$$ 

Analyze the running time of your algorithm.

(b) Give an efficient algorithm to print out such a sequence if one exists. Analyze the running time of your algorithm.