Problem 1. (1 point) Illustrate the operation of randomized quicksort on the array:
\[ A = (19, 2, 11, 14, 7, 17, 4, 3, 5, 15) \]
By showing the values in array \( A \) after each call to \texttt{partition}.

Solution:
\[
\begin{align*}
A &= [15, 2, 11, 14, 7, 17, 4, 3, 5, 19] \quad \# \text{Partitioned around 19} \\
A &= [5, 2, 11, 7, 4, 3, 14, 17, 15, 19] \quad \# \text{Partitioned around 14} \\
A &= [2, 5, 11, 7, 4, 3, 14, 17, 15, 19] \quad \# \text{Partitioned around 2} \\
A &= [2, 3, 4, 5, 11, 7, 14, 17, 15, 19] \quad \# \text{Partitioned around 5} \\
A &= [2, 3, 4, 5, 11, 7, 14, 17, 15, 19] \quad \# \text{Partitioned around 4} \\
A &= [2, 3, 4, 5, 7, 11, 14, 17, 15, 19] \quad \# \text{Partitioned around 7} \\
A &= [2, 3, 4, 5, 7, 11, 14, 15, 17, 19] \quad \# \text{Partitioned around 15}
\end{align*}
\]

Problem 2 (CLRS 7.2-5). (2 points) Suppose that the splits at every level of quicksort are in the proportion \( 1 - \alpha \) to \( \alpha \), where \( 0 < \alpha \leq 1/2 \) is a constant. Show that the minimum depth of a leaf in the recursion tree is approximately \( -\frac{\log n}{\log \alpha} \) and the maximum depth is approximately \( -\frac{\log n}{\log (1 - \alpha)} \).

Solution: With \( 0 < \alpha \leq 1/2 \) we have \( \alpha \leq 1 - \alpha \), indicating that \( \alpha \) corresponds to a greater or equal reduction in the problem size than \( 1 - \alpha \). Thus, the minimum depth of a leaf in the recursion tree will be observed along the path where the problem of size \( n \) is reduced to \( \alpha n \), and the maximum depth – along the path where the problem size is reduced to \( (1 - \alpha)n \).

We can view the length of each path as the number of times the initial problem size \( n \) can be divided by \( 1/\alpha \) and \( 1/(1 - \alpha) \) respectively before reaching the base case \( n = 1 \) (the problem statement allows us to ignore integer round-off):
\[
\begin{align*}
I_{\min} &\approx \log_{1/\alpha} n = \log_{\alpha^{-1}} n = \frac{\log n}{\log \alpha^{-1}} = \frac{\log n}{\log \alpha}, \\
I_{\max} &\approx \log_{1/(1-\alpha)} n = \log_{(1-\alpha)^{-1}} n = \frac{\log n}{\log (1-\alpha)^{-1}} = \frac{\log n}{\log(1-\alpha)}.
\end{align*}
\]

Problem 3 (CLRS 7.2-6). (3 points) Argue that for any constant \( 0 < \alpha \leq 1/2 \), the probability is approximately \( 1 - 2\alpha \) that on a random input array, \texttt{partition} produces a split more balanced than \( 1 - \alpha \) to \( \alpha \).

Solution: Let us rename the elements of the initial array as \( z_1, z_2, ..., z_n \) with \( z_i \) being the \( i \)-th smallest element. Observing that taking \( z_i \) as a pivot results in a split \( i : n - i \), and defining \( S_i \)
as the set of splits resulting from taking pivots from the set \( \{ z_i \mid i \in I \} \), we get:

\[
S_L = S_{\{1, \ldots, an\}} = \{1 : n - 1, \ldots, an : n - an\},
\]

\[
S_M = S_{\{an+1, \ldots, (1-\alpha)n-1\}} = \{an+1 : n - an - 1, \ldots, (1 - \alpha)n - 1 : n - (1 - \alpha)n + 1\},
\]

\[
S_H = S_{\{(1-\alpha)n, \ldots, n\}} = \{(1 - \alpha)n : n - (1 - \alpha)n, \ldots, n : 0\}.
\]

That is, the splits:

- Start with unbalanced \( 1 : n - 1 \) at \( z_1 \) and grow increasingly balanced.
- Become as balanced as \( \alpha : 1 - \alpha \) at \( z_{an} \).
- Grow increasingly balanced up to \( n/2 : n/2 \) at \( z_{n/2} \) and grow less balanced from there.
- Become as balanced as \( 1 - \alpha : \alpha \) at \( z_{(1-\alpha)n} \).
- End with unbalanced \( n : 0 \) at \( z_n \).

We then observe that only splits in \( S_M \) are more balanced than \( 1 - \alpha : \alpha \) and therefore the probability \( P_\alpha \) of getting a split more balanced than \( 1 - \alpha : \alpha \) is the probability that a pivot, selected uniformly at random among \( n \) elements, has index \( i \in M, M = \{an+1, \ldots, (1-\alpha)n-1\} \):

\[
P_\alpha = \frac{|M|}{n} = \frac{(1 - \alpha)n - 1 - (an + 1) + 1}{n} = \frac{n - 2an - 1}{n} = 1 - 2\alpha - \frac{1}{n} \approx 1 - 2\alpha \quad \text{for sufficiently large } n.
\]

**Problem 4 (CLRS 7.4-3).** (2 points) Show that the expression \( q^2 + (n - q - 1)^2 \) achieves a maximum over \( q = 0, 1, \ldots, n - 1 \) when \( q = 0 \) or \( q = n - 1 \).

**Solution:** Let us define the function \( f(q) \) and take the derivatives:

\[
f(q) = q^2 + (n - q - 1)^2,
\]

\[
f'(q) = 4q - 2n + 2,
\]

\[
f''(q) = 4.
\]

Setting the first derivative to zero and solving \( 4q - 2n + 2 = 0 \) shows that the function has a single stationary point at \( q = \frac{n - 1}{2} \). The positive second derivative shows that \( q = \frac{n - 1}{2} \) is a minimum, and thus \( f(q) \) can only achieve a maximum at the endpoints of the interval \([0, n-1]\). Evaluating the function at the endpoints confirms that \( f(q) \) achieves a maximum when \( q = 0 \) or \( q = n - 1 \):

\[
f(0) = (n - 1)^2,
\]

\[
f(n - 1) = (n - 1)^2.
\]
Problem 5 (CLRS 7.4-2). (3 points) Show that quicksort’s best-case running time is $\Omega(n \log n)$.

Solution: We can formulate the recurrence for quicksort’s best case by always considering a split resulting in a minimum running time:

$$T(n) = \min_{0 \leq q \leq n-1} (T(q) + T(n-q-1)) + \Theta(n).$$

Let us prove by substitution that $T(n) = \Omega(n \log n)$. We start with the base case $T(1) = \Theta(1)$, the inductive hypothesis:

$$T(k) \geq ck \log k \quad \text{for all } k: 1 < k < n,$$

and proceed with the inductive step:

$$T(n) = \min_{0 \leq q \leq n-1} (T(q) + T(n-q-1)) + \Theta(n) \geq \min_{0 \leq q \leq n-1} (cq \log q + c(n-q-1) \log(n-q-1)) + \Theta(n) = c \cdot \min_{0 \leq q \leq n-1} (q \log q + (n-q-1) \log(n-q-1)) + \Theta(n).$$

Let us define $f(q) = q \log q + (n-q-1) \log(n-q-1)$ and take the derivatives:

$$f'(q) = \frac{d}{dq} \left( q \frac{\log q + (n-q-1) \frac{\log(n-q-1)}{\log 2}}{\log 2} \right) = \frac{\left( \ln q + q \frac{1}{q} + (-1) \ln(n-q-1) + (n-q-1) \frac{1}{n-q-1} (-1) \right) \log 2}{\ln^2 2} = \frac{\ln q - \ln(n-q-1)}{\ln 2},$$

$$f''(q) = \frac{\ln 2 \left( \frac{1}{q} + \frac{1}{n-q-1} \right)}{\ln^2 2} = \frac{n-1}{q(n-q-1) \log 2}.$$ 

Setting the first derivative to zero gives a stationary point $q = \frac{n-1}{2}$:

$$\frac{\ln q - \ln(n-q-1)}{\ln 2} = 0 \quad \ln q - \ln(n-q-1) = 0 \quad \ln q = \ln(n-q-1) \quad q = n-q-1 \quad q = \frac{n-1}{2},$$

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in which the second derivative is positive for all \( n > 1 \):

\[
f'' \left( \frac{n-1}{2} \right) = \frac{n-1}{\frac{n-1}{2} \left( \frac{n-1}{2} - 1 \right) \ln 2}
\]

\[
= \frac{n-1}{\frac{(n-1)^2}{4} \ln 2}
\]

\[
= \frac{4}{(n-1) \ln 2}
\]

\[> 0 \quad \text{for all} \quad n > 1.
\]

Therefore, \( q = \frac{n-1}{2} \) is a minimum (we note that \( f(q) \) is undefined at both endpoints).

Returning to the substitution proof:

\[
T(n) \geq c \cdot \min_{0 \leq q \leq n-1} \left( q \lg q + (n-q-1) \lg(n-q-1) \right) + \Theta(n)
\]

\[
= c \left( \frac{n-1}{2} \lg \frac{n-1}{2} + \left( \frac{n-1}{2} - 1 \right) \lg \left( \frac{n-1}{2} - 1 \right) \right) + \Theta(n)
\]

\[
= c \left( \frac{n-1}{2} \lg \frac{n-1}{2} + \frac{n-1}{2} - \frac{n-1}{2} + \frac{n}{2} \right) + \Theta(n)
\]

\[
= c(n-1) \lg \frac{n-1}{2} + \Theta(n)
\]

\[
= c(n-1) \lg(n-1) - c(n-1) \lg 2 + \Theta(n)
\]

\[
= cn \lg(n-1) - c \lg(n-1) - c(n-1) + \Theta(n).
\]

As \( 1 < k < n \), we know that \( n \geq 3 \) and thus:

\[
T(n) \geq cn \lg(n-1) - c \lg(n-1) - c(n-1) + \Theta(n)
\]

\[
\geq cn \lg \frac{n}{2} - c \lg(n-1) - c(n-1) + \Theta(n)
\]

\[
= cn \lg n - cn - c \lg(n-1) - cn + c + \Theta(n)
\]

\[
= cn \lg n + (-2cn - c \lg(n-1) + c + \Theta(n)).
\]

Our inductive step holds as long as the residual is non-negative:

\[
-2cn - c \lg(n-1) + c + \Theta(n) \geq 0
\]

\[
\Theta(n) \geq c(2n + \lg(n-1) - 1),
\]

and we can always pick a sufficiently small \( c \) for the inequality to hold.

Having completed the substitution proof, we conclude that \( T(n) = \Omega(n \lg n) \).