Administration

• Class Web Site
  http://cs.nyu.edu/courses/summer16/CSCI-GA.2340-001/

• Mailing List
  Subscribe at see website
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• TA/Office Hours, etc: Jianbo Sun

• Homework
Contradiction

• Proof by contradiction
  – Suppose the statement to be proved is false
  – Show that this supposition leads logically to a contradiction
  – Conclude that the statement to be proved is true

• Square root of 2 is irrational

• There are infinite primes
Contraposition

- Proof by contraposition
  - Prepare the statement in the form: $\forall x \in D, P(x) \rightarrow Q(x)$
  - Rewrite this statement in the form: $\forall x \in D, \sim Q(x) \rightarrow \sim P(x)$
  - Prove the contrapositive by a direct proof

- Close relationship between proofs by contradiction and contraposition
Exercise

• Show that for integers \( n \), \( n^2 \) is odd if and only if \( n \) is odd
• Show that for all integers \( n \) and all prime numbers \( p \), if \( n^2 \) is divisible by \( p \), then \( n \) is divisible by \( p \)
• For all integers \( m \) and \( n \), if \( m+n \) is even then \( m \) and \( n \) are both even or \( m \) and \( n \) are both odd
• The product of any non-zero rational number and any irrational number is irrational
• If \( a \), \( b \), and \( c \) are integers and \( a^2+b^2=c^2 \), must at least one of \( a \) and \( b \) be even?
• Can you find two irrational numbers so that one raised to the power of another would produce a rational number?
Classic Number Theory Results

• Square root of 2 is irrational
• For any integer \(a\) and any integer \(k > 1\), if \(k \mid a\), then \(k\) does not divide \((a + 1)\)
• The set of prime numbers is infinite
Exercises

• Show that
  – a square of 3 is irrational
  – for any integer a, 4 does not divide \((a^2 - 2)\)
  – if \(n\) is not a perfect square then its square is irrational
  – \(\sqrt{2} + \sqrt{3}\) is irrational
  – \(\log_2(3)\) is irrational
  – every integer greater than 11 is a sum of two composite numbers
  – if \(p_1, p_2, \ldots, p_n\) are distinct prime numbers with \(p_1 = 2\), then \(p_1p_2\ldots p_n + 1\) has remainder 3 when divided by 4
  – for all integers \(n\), if \(n > 2\), then there exists prime number \(p\), such that \(n < p < n!\)
Basics of Set Theory

- Set and element are undefined notions in the set theory and are taken for granted.
- Set notation: \{1, 2, 3\}, \{\{1, 2\}, \{3\}, \{1, 2, 3\}\}, \{1, 2, 3, \ldots\}, \emptyset, \{x \in \mathbb{R} \mid -3 < x < 6\}
- Set A is called a subset of set B, written as \(A \subseteq B\), when \(\forall x, x \in A \rightarrow x \in B\). What is negation?
- A is a proper subset of B, when A is a subset of B and \(\exists x \in B \text{ and } x \notin A\)
- Visual representation of the sets
- Distinction between \(\subseteq\) and \(\in\)
Set Operations

- Set $A$ equals set $B$, iff every element of set $A$ is in set $B$ and vice versa. $(A = B \iff A \subseteq B \land B \subseteq A)$
- Proof technique for showing sets equality (example later for DeMorgan’s Law)
- Union of two sets is a set of all elements that belong to at least one of the sets (notation on board)
- Intersection of two sets is a set of all elements that belong to both sets (notation on board)
- Difference of two sets is a set of elements in one set, but not the other (notation on board)
- Complement of a set is a difference between universal set and a given set (notation on board)
- Examples
Empty Set

- \( S = \{x \in \mathbb{R}, x^2 = -1\} \)
- \( X = \{1, 3\}, \ Y = \{2, 4\}, \ C = X \cap Y \) (\(X\) and \(Y\) are disjoint)
- Empty set has no elements \(\emptyset\)
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:
  - \( A \cup \emptyset = A, \ A \cap \emptyset = \emptyset \)
  - \( A \cap A^c = \emptyset, \ A \cup A^c = U \)
  - \( U^c = \emptyset, \ \emptyset^c = U \)
Set Partitioning

• Two sets are called disjoint if they have no elements in common
• Theorem: $A - B$ and $B$ are disjoint
• A collection of sets $A_1, A_2, \ldots, A_n$ is called mutually disjoint when any pair of sets from this collection is disjoint
• A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is called a partition of a set $A$ when the union of these sets is $A$ and this collection consists of mutually disjoint sets
Power Set

• Power set of A is the set of all subsets of A
• Example on board
• Theorem: if $A \subseteq B$, then $P(A) \subseteq P(B)$
• Theorem: If set $X$ has $n$ elements, then $P(X)$ has $2^n$ elements
Cartesian Products

• Ordered n-tuple is a set of ordered n elements. Equality of n-tuples
• Cartesian product of n sets is a set of n-tuples, where each element in the n-tuple belongs to the respective set participating in the product
Set Properties

- **Inclusion of Intersection:**
  \[A \cap B \subseteq A\text{ and } A \cap B \subseteq B\]

- **Inclusion in Union:**
  \[A \subseteq A \cup B\text{ and } B \subseteq A \cup B\]

- **Transitivity of Inclusion:**
  \[(A \subseteq B \land B \subseteq C) \rightarrow A \subseteq C\]

- **Set Definitions:**
  \[x \in X \cup Y \iff x \in X \lor y \in Y\]
  \[x \in X \cap Y \iff x \in X \land y \in Y\]
  \[x \in X - Y \iff x \in X \land y \notin Y\]
  \[x \in X^c \iff x \notin X\]
  \[(x, y) \in X \times Y \iff x \in X \land y \in Y\]
Set Identities

• Commutative Laws: \( A \cap B = A \cap B \) and \( A \cup B = B \cup A \)
• Associative Laws: \( (A \cap B) \cap C = A \cap (B \cap C) \) and \( (A \cup B) \cup C = A \cup (B \cup C) \)
• Distributive Laws:
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \] and
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
• Intersection and Union with universal set: \( A \cap U = A \) and \( A \cup U = U \)
• Double Complement Law: \( (A^c)^c = A \)
• Idempotent Laws: \( A \cap A = A \) and \( A \cup A = A \)
• De Morgan’s Laws: \( (A \cap B)^c = A^c \cup B^c \) and \( (A \cup B)^c = A^c \cap B^c \)
• Absorption Laws: \( A \cup (A \cap B) = A \) and \( A \cap (A \cup B) = A \)
• Alternate Representation for Difference: \( A - B = A \cap B^c \)
• Intersection and Union with a subset: if \( A \subseteq B \), then \( A \cap B = A \) and \( A \cup B = B \)
Proving Equality

• First show that one set is a subset of another
• To show this, choose an arbitrary particular element as with direct proofs (call it x), and show that if x is in A then x is in B to show that A is a subset of B
• Example for DeMorgan’s (step through all cases)
Disproofs, Counterexamples and Algebraic Proofs

• Is it true that \((A - B) \cup (B - C) = A - C\)? (No via counterexample)

• Show that \((A \cup B) - C = (A - C) \cup (B - C)\) (Can do with an algebraic proof, slightly different)
Russell’s Paradox

• Set of all integers, set of all abstract ideas
• Consider $S = \{ A, A \text{ is a set and } A \notin A\}$
• Is $S$ an element of $S$?
• Barber puzzle: a male barber shaves all those men who do not shave themselves. Does the barber shave himself?
• Consider $S = \{ A \subseteq U, A \notin A\}$. Is $S \in S$?
• Godel: No way to rigorously prove that mathematics is free of contradictions. (“This statement is not provable” is true but not provable) (consistency of an axiomatic system is not provable within that system)
Halting Problem

- There is no computer algorithm that will accept any algorithm X and data set D as input and then will output “halts” or “loops forever” to indicate whether X terminates in a finite number of steps when X is run with data set D.

- Proof is by contradiction (Read this pg 222, and we will review later)
Generic Functions

• A function $f: X \rightarrow Y$ is a relationship between elements of $X$ to elements of $Y$, when each element from $X$ is related to a unique element from $Y$

• $X$ is called domain of $f$, range of $f$ is a subset of $Y$ so that for each element $y$ of this subset there exists an element $x$ from $X$ such that $y = f(x)$

• Sample functions:
  – $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$
  – $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x + 1$
  – $f : \mathbb{Q} \rightarrow \mathbb{Z}$, $f(x) = 2$
Generic Functions

• Arrow diagrams for functions
• Non-functions
• Equality of functions:
  – $f(x) = |x|$ and $g(x) = \sqrt{x^2}$
• Identity function
• Logarithmic function
One-to-One Functions

• Function $f : X \rightarrow Y$ is called one-to-one (injective) when for all elements $x_1$ and $x_2$ from $X$ if $f(x_1) = f(x_2)$, then $x_1 = x_2$

• Determine whether the following functions are one-to-one:
  – $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 4x - 1$
  – $g : \mathbb{Z} \rightarrow \mathbb{Z}, g(n) = n^2$

• Hash functions
Onto Functions

• Function $f : X \rightarrow Y$ is called onto (surjective) when given any element $y$ from $Y$, there exists $x$ in $X$ so that $f(x) = y$

• Determine whether the following functions are onto:
  – $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 4x - 1$
  – $f : \mathbb{Z} \rightarrow \mathbb{Z}, g(n) = 4n - 1$

• Bijection is one-to-one and onto

• Reversing strings function is bijective
Inverse Functions

- If $f : X \rightarrow Y$ is a bijective function, then it is possible to define an inverse function $f^{-1} : Y \rightarrow X$ so that $f^{-1}(y) = x$ whenever $f(x) = y$
- Find an inverse for the following functions:
  - String-reverse function
  - $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 4x - 1$
- Inverse function of a bijective function is a bijective function itself
Composition of Functions

• Let \( f : X \to Y \) and \( g : Y \to Z \), let range of \( f \) be a subset of the domain of \( g \). The we can define a composition of \( g \circ f : X \to Z \)

• Let \( f,g : Z \to Z \), \( f(n) = n + 1 \), \( g(n) = n^2 \). Find \( f \circ g \) and \( g \circ f \). Are they equal?

• Composition with identity function

• Composition with an inverse function

• Composition of two one-to-one functions is one-to-one

• Composition of two onto functions is onto
Pigeonhole Principle

• If $n$ pigeons fly into $m$ pigeonholes and $n > m$, then at least one hole must contain two or more pigeons.

• A function from one finite set to a smaller finite set cannot be one-to-one.

• In a group of 13 people must there be at least two who have birthday in the same month?

• A drawer contains 10 black and 10 white socks. How many socks need to be picked to ensure that a pair is found?

• Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If 5 integers are selected must at least one pair have sum of 9?
Pigeonhole Principle

• Generalized Pigeonhole Principle: For any function $f : X \rightarrow Y$ acting on finite sets, if $n(X) > k \times N(Y)$, then there exists some $y$ from $Y$ so that there are at least $k + 1$ distinct $x$’s so that $f(x) = y$

• “If $n$ pigeons fly into $m$ pigeonholes, and, for some positive $k$, $m > k \times m$, then at least one pigeonhole contains $k+1$ or more pigeons”

• In a group of 85 people at least 4 must have the same last initial.

• There are 42 students who are to share 12 computers. Each student uses exactly 1 computer and no computer is used by more than 6 students. Show that at least 5 computers are used by 3 or more students.
Cardinality

- Cardinality refers to the size of the set
- Finite and infinite sets
- Two sets have the same cardinality when there is bijective function associating them
- Cardinality is reflexive, symmetric and transitive
- Countable sets: set of all integers, set of even numbers, positive rationals (Cantor diagonalization)
- Set of real numbers between 0 and 1 has same cardinality as set of all reals
- Computability of functions