Administration

• Class Web Site
  http://cs.nyu.edu/courses/summer16/CSCI-GA.2340-001/

• Mailing List
  Subscribe at see website
  Messages to: see website

• TA/Office Hours, etc: Jianbo Sun

• Homework
Conditional Statements

• Write truth table for: \( p \land q \rightarrow \neg p \)
• Show that \((p \lor q) \rightarrow r = (p \rightarrow r) \land (q \rightarrow r)\)
• Representation of \( \rightarrow \): \( p \rightarrow q = \neg p \lor q \)
• Re-write using if-else: Either you get in class on time, or you risk missing some material
• Negation of \( \rightarrow \): \( \neg(p \rightarrow q) = p \land \neg q \)
• Write negation for: If it is raining, then I cannot go to the beach
Conditional Statements

• Contrapositive $p \rightarrow q$ is another conditional statement $\sim q \rightarrow \sim p$

• A conditional statement is equivalent to its contrapositive

• The converse of $p \rightarrow q$ is $q \rightarrow p$

• The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$

• Conditional statement and its converse are not equivalent

• Conditional statement and its inverse are not equivalent
Conditional Statements

• The converse and the inverse of a conditional statement are equivalent to each other.

• \( p \) only if \( q \) means \( \neg q \rightarrow \neg p \), or \( p \rightarrow q \).

• Biconditional of \( p \) and \( q \) means “\( p \) if and only if \( q \)” and is denoted as \( p \iff q \).

• \( r \) is a sufficient condition for \( s \) means “if \( r \) then \( s \)”.

• \( r \) is a necessary condition for \( s \) means “if not \( r \) then not \( s \)”.
Exercises

• Write contrapositive, converse and inverse statements for:
  – If $P$ is a square, then $P$ is a rectangle
  – If today is Thanksgiving, then tomorrow is Friday
  – If $c$ is rational, then the decimal expansion of $r$ is repeating
  – If $n$ is prime, then $n$ is odd or $n$ is 2
  – If $x$ is nonnegative, then $x$ is positive or $x$ is 0
  – If Tom is Ann’s father, then Jim is her uncle and Sue is her aunt
  – If $n$ is divisible by 6, then $n$ is divisible by 2 and $n$ is divisible by 3
Arguments

• An argument is a sequence of statements. All statements except the final one are called premises (or assumptions or hypotheses). The final statement is called the conclusion.

• An argument is considered valid if from the truth of all premises, the conclusion must also be true.

• The conclusion is said to be inferred or deduced from the truth of the premises.
Arguments

• Test to determine the validity of the argument:
  – Identify the premises and conclusion of the argument
  – Construct the truth table for all premises and the conclusion
  – Find critical rows in which all the premises are true
  – If the conclusion is true in all critical rows then the argument is valid, otherwise it is invalid

• Example of valid argument form:
  – Premises: $p \lor (q \lor r)$ and $\neg r$, conclusion: $p \lor q$

• Example of invalid argument form:
  – Premises: $p \rightarrow q \lor \neg r$ and $q \rightarrow p \land r$, conclusion: $p \rightarrow r$
Valid Argument-Forms

• Modus ponens (method of affirming):
  – Premises: \( p \rightarrow q \) and \( p \), conclusion: \( q \)

• Modus tollens (method of denying):
  – Premises: \( p \rightarrow q \) and \( \sim q \), conclusion: \( \sim p \)

• Disjunctive addition:
  – Premises: \( p \), conclusion: \( p \mid q \)
  – Premises: \( q \), conclusion: \( p \mid q \)

• Conjunctive simplification:
  – Premises: \( p \& q \), conclusion: \( p, q \)
Valid Argument-Forms

• Disjunctive Syllogism:
  – Premises: $p \lor q$ and $\neg q$, conclusion: $p$
  – Premises: $p \lor q$ and $\neg p$, conclusion: $q$

• Hypothetical Syllogism
  – Premises: $p \rightarrow q$ and $q \rightarrow r$, conclusion: $p \rightarrow r$

• Dilemma: proof by division into cases:
  – Premises: $p \lor q$ and $p \rightarrow r$ and $q \rightarrow r$, conclusion: $r$
Complex Deduction

• Premises:
  – If my glasses are on the kitchen table, then I saw them at breakfast
  – I was reading the newspaper in the living room or I was reading the newspaper in the kitchen
  – If I was reading the newspaper in the living room, then my glasses are on the coffee table
  – I did not see my glasses at breakfast
  – If I was reading my book in bed, then my glasses are on the bed table
  – If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table

• Where are the glasses?
Fallacies

• A fallacy is an error in reasoning that results in an invalid argument

• Three common fallacies:
  – Vague or ambiguous premises
  – Begging the question (assuming what is to be proved)
  – Jumping to conclusions without adequate grounds

• Converse Error:
  – Premises: $p \rightarrow q$ and $q$, conclusion: $p$

• Inverse Error:
  – Premises: $p \rightarrow q$ and $\neg p$, conclusion: $\neg q$
Fallacies

- It is possible for a valid argument to have false conclusion and for an invalid argument to have a true conclusion:
  - Premises: if John Lennon was a rock star, then John Lennon had red hair, John Lennon was a rock star; Conclusion: John Lennon had red hair
  - Premises: If New York is a big city, then New York has tall buildings, New York has tall buildings; Conclusion: New York is a big city
Contradiction

- Contradiction rule: if one can show that the supposition that a statement $p$ is false leads to a contradiction, then $p$ is true.
- Knight is a person who always says truth, knave is a person who always lies:
  - A says: B is a knight
  - B says: A and I are of opposite types
What are A and B?
Digital Logic Circuits

- Digital Logic Circuit is a basic electronic component of a digital system
- Values of digital signals are 0 or 1 (bits)
- Black Box is specified by the signal input/output table
- Three gates: NOT-gate, AND-gate, OR-gate
- Combinational circuit is a combination of logical gates
- Combinational circuit always correspond to some boolean expression, such that input/output table of a table and a truth table of the expression are identical
Number Systems

- Decimal number system
- Binary number system
- Conversion between decimal and binary numbers
- Binary addition and subtraction
Logic of Quantified Statements
Predicates

• A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

• The domain of a predicate variable is a set of all values that may be substituted in place of the variable.

• \( P(x) \): \( x \) is a student at NYU.
Predicates

• If $P(x)$ is a predicate and $x$ has domain $D$, the truth set of $P(x)$ is the set of all elements in $D$ that make $P(x)$ true when substituted for $x$. The truth set is denoted as:

  $$\{x \in D \mid P(x)\}$$

• Let $P(x)$ and $Q(x)$ be predicates with the common domain $D$. $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$. $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets.
Universal Quantifier

• Let $P(x)$ be a predicate with domain $D$. A universal statement is a statement in the form “$\forall x \in D, P(x)$”. It is true iff $P(x)$ is true for every $x$ from $D$. It is false iff $P(x)$ is false for at least one $x$ from $D$. A value of $x$ from which $P(x)$ is false is called a counterexample to the universal statement.

• Examples
  – $D = \{1, 2, 3, 4, 5\}$: $\forall x \in D, x^2 \geq x$
  – $\forall x \in \mathbb{R}, x^2 \geq x$

• Method of exhaustion
Existential Quantifier

• Let \( P(x) \) be a predicate with domain \( D \). An existential statement is a statement in the form “\( \exists x \in D, P(x) \)”. It is true iff \( P(x) \) is true for at least one \( x \) from \( D \). It is false iff \( P(x) \) is false for every \( x \) from \( D \).

• Examples:
  – \( \exists m \in \mathbb{Z}, m^2 = m \)
  – \( E = \{5, 6, 7, 8, 9\}, \exists x \in E, m^2 = m \)
Universal Conditional Statement

• Universal conditional statement “∀x, if P(x) then Q(x)”:
  – ∀x R, if x > 2, then x² > 4

• Writing Conditional Statements Formally

• Universal conditional statement is called vacuously true or true by default iff P(x) is false for every x in D
Negation of Quantified Statements

• The negation of a universally quantified statement \( \forall x \in D, P(x) \) is \( \exists x \in D, \neg P(x) \)

• “All balls in the bowl are red” – Vacuously True

Example for Universal Statements

• The negation of an existentially quantified statement \( \exists x \in D, P(x) \) is \( \forall x \in D, \neg P(x) \)

• The negation of a universal conditional statement \( \forall x \in D, P(x) \rightarrow Q(x) \) is \( \exists x \in D, P(x) \land \neg Q(x) \)
Exercises

• Write negations for each of the following statements:
  – All dinosaurs are extinct
  – No irrational numbers are integers
  – Some exercises have answers
  – All COBOL programs have at least 20 lines
  – The sum of any two even integers is even
  – The square of any even integer is even

• Let P(x) be some predicate defined for all real numbers x, let:
  \[ r = \forall x \in \mathbb{Z}, P(x); \quad s = \forall x \in \mathbb{Q}, P(x); \quad t = \forall x \in \mathbb{R}, P(x) \]
  – Find P(x) (but not \( x \in \mathbb{Z} \)) so that \( r \) is true, but \( s \) and \( t \) are false
  – Find P(x) so that both \( r \) and \( s \) are true, but \( t \) is false
Variants of Conditionals

• Contrapositive
• Converse
• Inverse
• Generalization of relationships from before
• Examples
Necessary and Sufficient Conditions, Only If

- $\forall x, r(x)$ is a sufficient condition for $s(x)$ means: $\forall x, \text{if } r(x) \text{ then } s(x)$
- $\forall x, r(x)$ is a necessary condition for $s(x)$ means: $\forall x, \text{if } s(x) \text{ then } r(x)$ (or $\forall x, \text{if } \neg r(x) \text{ then } \neg s(x)$, if $r(x)$ does not happen, $s(x)$ cannot happen)
- $\forall x, r(x)$ only if $s(x)$ means: $\forall x, \text{if } r(x) \text{ then } s(x)$ (or $\forall x, \text{if } \neg s(x) \text{ then } \neg r(x)$)
Multiply Quantified Statements

- For all positive numbers $x$, there exists number $y$ such that $y < x$
- There exists number $x$ such that for all positive numbers $y$, $y < x$
- For all people $x$ there exists person $y$ such that $x$ loves $y$
- There exists person $x$ such that for all people $y$, $x$ loves $y$
- Definition of mathematical limit:
  \[
  \lim_{x \to c} f(x) = L \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x-c| < \delta \Rightarrow |f(x)-L| < \varepsilon)
  \]
- Order of quantifiers matters in some (most) cases (will find page reference, “lively discussion”):
  http://math.stackexchange.com/questions/201051/is-the-order-of-universal-existential-quantifiers-important
Negation of Multiply Quantified Statements

• The negation of $\forall x, \exists y, P(x, y)$ is logically equivalent to $\exists x, \forall y, \neg P(x, y)$

• The negation of $\exists x, \forall y, P(x, y)$ is logically equivalent to $\forall x, \exists y, \neg P(x, y)$
Prolog Programming Language

• Can use parts of logic as programming lang.

• Simple statements:
  isabove(g, b), color(g, gray)

• Quantified statements:
  if isabove(X, Y) and isabove(Y, Z) then isabove(X, Z)

• Questions:
  ?color(b, blue), ?isabove(X, w)
Exercises

• Determine whether a pair of quantified statements have the same truth values
  - \( \forall x \in D, (P(x) \land Q(x)) \) vs \( (\forall x \in D, P(x)) \land (\forall x \in D, Q(x)) \)
  - \( \exists x \in D, (P(x) \land Q(x)) \) vs \( (\exists x \in D, P(x)) \land (\exists x \in D, Q(x)) \)
  - \( \forall x \in D, (P(x) \lor Q(x)) \) vs \( (\forall x \in D, P(x)) \lor (\forall x \in D, Q(x)) \)
  - \( \exists x \in D, (P(x) \lor Q(x)) \) vs \( (\exists x \in D, P(x)) \lor (\exists x \in D, Q(x)) \)
Arguments with Quantified Statements

• Rule of universal instantiation: if some property is true of everything in the domain, then this property is true for any subset in the domain

• Universal Modus Ponens:
  – Premises: (∀x, if P(x) then Q(x)); P(a) for some a
  – Conclusion: Q(a)

• Universal Modus Tollens:
  – Premises: (∀x, if P(x) then Q(x)); ¬Q(a) for some a
  – Conclusion: ¬P(a)

• Converse and inverse errors
Validity of Arguments using Diagrams

• Premises: All human beings are mortal; Zeus is not mortal. Conclusion: Zeus is not a human being

• Premises: All human beings are mortal; Felix is mortal. Conclusion: Felix is a human being

• Premises: No polynomial functions have horizontal asymptotes; This function has a horizontal asymptote. Conclusion: This function is not a polynomial
Proof and Counterexample

- Discovery and proof
- Even and odd numbers
  - number \( n \) from \( \mathbb{Z} \) is called even if \( \exists k \in \mathbb{Z}, n = 2k \)
  - number \( n \) from \( \mathbb{Z} \) is called odd if \( \exists k \in \mathbb{Z}, n = 2k + 1 \)
- Prime and composite numbers
  - number \( n \) from \( \mathbb{Z} \) is called prime if
    \[ \forall r, s \in \mathbb{Z}, n = r \times s \rightarrow r = 1 \lor s = 1 \]
  - number \( n \) from \( \mathbb{Z} \) is called composite if
    \[ \exists r, s \in \mathbb{Z}, n = r \times s \land r > 1 \land s > 1 \]
Proving Statements

• Constructive proofs for existential statements
• Example: Show that there is a prime number that can be written as a sum of two perfect squares
• Universal statements: method of exhaustion and generalized proof
• Direct Proof:
  – Express the statement in the form: \( \forall x \in D, P(x) \rightarrow Q(x) \)
  – Take an arbitrary \( x \) from \( D \) so that \( P(x) \) is true
  – Show that \( Q(x) \) is true based on previous axioms, theorems, \( P(x) \) and rules of valid reasoning
Proof

• Show that if the sum of any two integers is even, then so is their difference

• Common mistakes in a proof
  – Arguing from example
  – Using the same symbol for different variables
  – Jumping to a conclusion
  – Begging the question
Counterexample

- To show that the statement in the form “∀x ∈ D, P(x) → Q(x)” is not true one needs to show that the negation, which has a form “∃x ∈ D, P(x) ∧ ~Q(x)” is true. x is called a counterexample.

- Famous conjectures:
  - Fermat big theorem: there are no non-zero integers x, y, z such that x^n + y^n = z^n, for n > 2
  - Goldbach conjecture: any even integer can be represented as a sum of two prime numbers
  - Euler’s conjecture: no three perfect fourth powers add up to another perfect fourth power
Exercises

• Any product of four consecutive positive integers is one less than a perfect square
• To check that an integer is a prime it is sufficient to check that \( n \) is not divisible by any prime less than or equal to \( \sqrt{n} \)
• If \( p \) is a prime, is \( 2^p - 1 \) a prime too?
• Does \( 15x^3 + 7x^2 - 8x - 27 \) have an integer zero?