Sequences

• Sequence is a set of (usually infinite number of) ordered elements: \(a_1, a_2, \ldots, a_n, \ldots\)
• Each individual element \(a_k\) is called a term, where \(k\) is called an index
• Sequences can be computed using an explicit formula: \(a_k = k \times (k + 1)\) for \(k > 1\)
• Alternate sign sequences
• Finding an explicit formula given initial terms of the sequence: 1, -1/4, 1/9, -1/16, 1/25, -1/36, …
• Sequence is (most often) represented in a computer program as a single-dimensional array
Sequence Operations

• Summation: $\sum$, expanded form, limits (lower, upper) of summation, dummy index

• Change of index inside summation

• Product: $\prod$, expanded form, limits (lower, upper) of product, dummy index

• Factorial: $n!$, $n! = n \times (n - 1)!$
Sequences

• Geometric sequence:  
a, ar, ar^2, ar^3, ..., ar^n

• Arithmetic sequence:  
a, a+d, a +2d, ..., a+nd

• Sum of geometric sequence:  
\[ \sum_{0 \rightarrow n} ar^k \]

• Sum of arithmetic sequence:  
\[ \sum_{0 \rightarrow n} a+kd \]
Review Mathematical Induction

• Principle of Mathematical Induction:
  Let \( P(n) \) be a predicate that is defined for integers \( n \) and let \( a \) be some integer. If the following two premises are true:
  \begin{align*}
  P(a) & \text{ is a true} \\
  \forall k \geq a, \ P(k) \rightarrow P(k + 1)
  \end{align*}
then the following conclusion is true as well
  \( P(n) \) is true for all \( n \geq a \)
Applications of Mathematical Induction

• Show that \(1 + 2 + \ldots + n = n * (n + 1) / 2\) (Prove on board)

• Sum of geometric series:
  \[r^0 + r^1 + \ldots + r^n = (r^{n+1} - 1) / (r - 1)\] (Prove on board)
Examples that Can be Proved with Mathematical Induction

- Show that $2^{2n} - 1$ is divisible by 3 (in book)
- Show (on board) that for $n > 2$: $2n + 1 < 2^n$
- Show that $x^n - y^n$ is divisible by $x - y$
- Show that $n^3 - n$ is divisible by 6 (similar to book problem)
Strong Mathematical Induction

• Utilization of predicates $P(a)$, $P(a + 1)$, ..., $P(n)$ to show $P(n + 1)$.
• Variation of normal M.I., but basis may contain several proofs and in assumption, truth assumed for all values through from base to $k$.
• Examples:
  • Any integer greater than 1 is divisible by a prime
  • Existence and Uniqueness of binary integer representation (Read in book)
Well-Ordering Principle

- Well-ordering principle for integers: a set of integers that are bounded from below (all elements are greater than a fixed integer) contains a least element
- Example:
- Existence of quotient-remainder representation of an integer $n$ against integer $d$
Counting and Probability

• Coin tossing
• Random process
• Sample space is the set of all possible outcomes of a random process. An event is a subset of a sample space
• Probability of an event is the ratio between the number of outcomes that satisfy the event to the total number of possible outcomes
  \[ P(E) = \frac{N(E)}{N(S)} \]
  for event E and sample space S
• Rolling a pair of dice and card deck as sample random processes
Possibility Trees

• Teams A and B are to play each other repeatedly until one wins two games in a row or a total three games.
  – What is the probability that five games will be needed to determine the winner?

• Suppose there are 4 I/O units and 3 CPUs. In how many ways can I/Os and CPUs be attached to each other when there are no restrictions?
Multiplication Rule

- Multiplication rule: if an operation consists of $k$ steps each of which can be performed in $n_i$ ways ($i = 1, 2, \ldots, k$), then the entire operation can be performed in $\prod n_i$ ways.
- Number of PINs
- Number of elements in a Cartesian product
- Number of PINs without repetition
- Number of Input/Output tables for a circuit with $n$ input signals
- Number of iterations in nested loops
Multiplication Rule

• Three officers – a president, a treasurer and a secretary are to be chosen from four people: Alice, Bob, Cindy and Dan. Alice cannot be a president, Either Cindy or Dan must be a secretary. How many ways can the officers be chosen?
Permutations

A permutation of a set of objects is an ordering of these objects.

The number of permutations of a set of \( n \) objects is \( n! \) (Examples).

An \( r \)-permutation of a set of \( n \) elements is an ordered selection of \( r \) elements taken from a set of \( n \) elements: \( P(n, r) \) (Examples).

\[ P(n, r) = \frac{n!}{(n - r)!} \]

Show that \( P(n, 2) + P(n, 1) = n^2 \).
Addition Rule

• If a finite set \( A \) is a union of \( k \) mutually disjoint sets \( A_1, A_2, \ldots, A_k \), then \( n(A) = \sum n(A_i) \)

• Number of words of length no more than 3

• Number of 3-digit integers divisible by 5
Difference Rule

- If \( A \) is a finite set and \( B \) is its subset, then \( n(A - B) = n(A) - n(B) \)
- How many PINS contain repeated symbols?
- So, \( P(A^c) = 1 - P(A) \) (Example for PINS)
- How many students are needed so that the probability of two of them having the same birthday equals 0.5?
Inclusion/Exclusion Rule

- 2 sets
- 3 sets
Combinations

• An r-combination of a set of n elements is a subset of r elements: $C(n, r)$

• Permutation is an ordered selection, combination is an unordered selection

• Quantitative relationship between permutations and combinations: $P(n, r) = C(n, r) \times r!$

• Permutations of a set with repeated elements

• Double counting
Team Selection Problems

- There are 12 people, 5 men and 7 women, to work on a project:
  - How many 5-person teams can be chosen?
  - If two people insist on working together (or not working at all), how many 5-person teams can be chosen?
  - If two people insist on not working together, how many 5-person teams can be chosen?
  - How many 5-person teams consist of 3 men and 2 women?
  - How many 5-person teams contain at least 1 man?
  - How many 5-person teams contain at most 1 man?
Poker Problems

• What is a probability to contain one pair?
• What is a probability to contain two pairs?
• What is a probability to contain a triple?
• What is a probability to contain royal flush?
• What is a probability to contain straight flush?
• What is a probability to contain straight?
• What is a probability to contain flush?
• What is a probability to contain full house?
Combinations with Repetition

- An \( r \)-combination with repetition allowed is an unordered selection of elements where some elements can be repeated.
- The number of \( r \)-combinations with repetition allowed from a set of \( n \) elements is \( C(r + n - 1, r) \).
- Soft drink example
Algebra of Combinations and Pascal’s Triangle

• The number of \( r \)-combinations from a set of \( n \) elements equals the number of \( (n - r) \)-combinations from the same set.

• Pascal’s triangle: \( C(n + 1, r) = C(n, r - 1) + C(n, r) \)

• \( C(n, r) = C(n, n-r) \)
Probability Axioms

• $P(A^c) = 1 - P(A)$

• $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  – What if $A$ and $B$ mutually disjoint?
  (Then $P(A \cap B) = 0$)
Conditional Probability

- For events A and B in sample space S if $P(A) \neq 0$, then the probability of B given A is:
  $P(A \mid B) = \frac{P(A \cap B)}{P(A)}$

- Example with Urn and Balls:
  - An urn contains 5 blue and
Conditional Probability Example

• An urn contains 5 blue and 7 gray balls. 2 are chosen at random.
  - What is the probability they are blue?
  - Probability first is not blue but second is?
  - Probability second ball is blue?
  - Probability at least one ball is blue?
  - Probability neither ball is blue?
Conditional Probability Extended

• Imagine one urn contains 3 blue and 4 gray balls and a second urn contains 5 blue and 3 gray balls

• Choose an urn randomly and then choose a ball.

• What is the probability that if the ball is blue that it came from the first urn?
Bayes’ Theorem

• Extended version of last example.
• If S, our sample space, is the union of n mutually disjoint events, B1, B2, …, Bn and A is an even in S with \( P(A) \neq 0 \) and \( k \) is an integer between 1 and n, then:

\[
P(B_k | A) = \frac{P(A | B_k) \cdot P(B_k)}{P(A | B_1) \cdot P(B_1) + \ldots + P(A | B_n) \cdot P(B_n)}.
\]

Application: Medical Tests (false positives, etc.)
Independent Events

• If A and B are independent events, 
  \[ P(A \cap B) = P(A) \times P(B) \]

• If C is also independent of A and B
  \[ P(A \cap B \cap C) = P(A) \times P(B) \times P(C) \]

• Difference from Conditional Probability can be seen via Russian Roulette example.