Administration

• Class Web Site
  http://cs.nyu.edu/courses/summer15/CSCI-GA.2340-001/

• Mailing List
  Subscribe at TBA
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• TA/Office Hours, etc

• Homework
Arguments

• An argument is a sequence of statements. All statements except the final one are called premises (or assumptions or hypotheses). The final statement is called the conclusion.

• An argument is considered valid if from the truth of all premises, the conclusion must also be true.

• The conclusion is said to be inferred or deduced from the truth of the premises
Arguments

• Test to determine the validity of the argument:
  – Identify the premises and conclusion of the argument
  – Construct the truth table for all premises and the conclusion
  – Find critical rows in which all the premises are true
  – If the conclusion is true in all critical rows then the argument is valid, otherwise it is invalid

• Example of valid argument form:
  – Premises: p ∨ (q ∨ r) and ∼r, conclusion: p ∨ q

• Example of invalid argument form:
  – Premises: p → q ∨ ∼r and q → p ∧ r, conclusion: p → r
Valid Argument-Forms

• Modus ponens (method of affirming):
  – Premises: \( p \rightarrow q \) and \( p \), conclusion: \( q \)

• Modus tollens (method of denying):
  – Premises: \( p \rightarrow q \) and \( \neg q \), conclusion: \( \neg p \)

• Disjunctive addition:
  – Premises: \( p \), conclusion: \( p \mid q \)
  – Premises: \( q \), conclusion: \( p \mid q \)

• Conjunctive simplification:
  – Premises: \( p \& q \), conclusion: \( p, q \)
Valid Argument-Forms

• Disjunctive Syllogism:
  – Premises: p | q and ~q, conclusion: p
  – Premises: p | q and ~p, conclusion: q

• Hypothetical Syllogism
  – Premises: p → q and q → r, conclusion: p → r

• Dilemma: proof by division into cases:
  – Premises: p | q and p → r and q → r, conclusion: r
Complex Deduction

• Premises:
  – If my glasses are on the kitchen table, then I saw them at breakfast
  – I was reading the newspaper in the living room or I was reading the newspaper in the kitchen
  – If I was reading the newspaper in the living room, then my glasses are on the coffee table
  – I did not see my glasses at breakfast
  – If I was reading my book in bed, then my glasses are on the bed table
  – If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table

• Where are the glasses?
Fallacies

• A fallacy is an error in reasoning that results in an invalid argument

• Three common fallacies:
  – Vague or ambiguous premises
  – Begging the question (assuming what is to be proved)
  – Jumping to conclusions without adequate grounds

• Converse Error:
  – Premises: $p \rightarrow q$ and $q$, conclusion: $p$

• Inverse Error:
  – Premises: $p \rightarrow q$ and $\sim p$, conclusion: $\sim q$
Fallacies

• It is possible for a valid argument to have false conclusion and for an invalid argument to have a true conclusion:
  – Premises: if John Lennon was a rock star, then John Lennon had red hair, John Lennon was a rock star; Conclusion: John Lennon had red hair
  – Premises: If New York is a big city, then New York has tall buildings, New York has tall buildings; Conclusion: New York is a big city
Contradiction

• Contradiction rule: if one can show that the supposition that a statement $p$ is false leads to a contradiction, then $p$ is true.

• Knight is a person who always says truth, knave is a person who always lies:
  – A says: B is a knight
  – B says: A and I are of opposite types
What are A and B?
Digital Logic Circuits

- Digital Logic Circuit is a basic electronic component of a digital system
- Values of digital signals are 0 or 1 (bits)
- Black Box is specified by the signal input/output table
- Three gates: NOT-gate, AND-gate, OR-gate
- Combinational circuit is a combination of logical gates
- Combinational circuit always correspond to some boolean expression, such that input/output table of a table and a truth table of the expression are identical
Number Systems

• Decimal number system
• Binary number system
• Conversion between decimal and binary numbers
• Binary addition and subtraction
Logic of Quantified Statements
Predicates

- A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
- The domain of a predicate variable is a set of all values that may be substituted in place of the variable.
- $P(x)$: $x$ is a student at NYU.
Predicates

• If $P(x)$ is a predicate and $x$ has domain $D$, the truth set of $P(x)$ is the set of all elements in $D$ that make $P(x)$ true when substituted for $x$. The truth set is denoted as:

$$\{x \in D \mid P(x)\}$$

• Let $P(x)$ and $Q(x)$ be predicates with the common domain $D$. $P(x) \implies Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$. $P(x) \iff Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets.
Universal Quantifier

• Let $P(x)$ be a predicate with domain $D$. A universal statement is a statement in the form “$\forall x \in D, P(x)$”. It is true iff $P(x)$ is true for every $x$ from $D$. It is false iff $P(x)$ is false for at least one $x$ from $D$. A value of $x$ from which $P(x)$ is false is called a counterexample to the universal statement.

• Examples
  – $D = \{1, 2, 3, 4, 5\}$: $\forall x \in D, x^2 \geq x$
  – $\forall x \in R, x^2 \geq x$

• Method of exhaustion
Existential Quantifier

• Let $P(x)$ be a predicate with domain $D$. An existential statement is a statement in the form $\exists x \in D, P(x)$. It is true iff $P(x)$ is true for at least one $x$ from $D$. It is false iff $P(x)$ is false for every $x$ from $D$.

• Examples:
  - $\exists m \in \mathbb{Z}, m^2 = m$
  - $E = \{5, 6, 7, 8, 9\}, \exists x \in E, m^2 = m$
Universal Conditional Statement

• Universal conditional statement “∀x, if P(x) then Q(x)”: 
  – ∀x R, if x > 2, then x² > 4

• Writing Conditional Statements Formally

• Universal conditional statement is called vacuously true or true by default iff P(x) is false for every x in D
Negation of Quantified Statements

- The negation of a universally quantified statement $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$
- “All balls in the bowl are red” – Vacuously True Example for Universal Statements
- The negation of an existentially quantified statement $\exists x \in D, P(x)$ is $\forall x \in D, \neg P(x)$
- The negation of a universal conditional statement $\forall x \in D, P(x) \rightarrow Q(x)$ is $\exists x \in D, P(x) \land \neg Q(x)$
Exercises

• Write negations for each of the following statements:
  – All dinosaurs are extinct
  – No irrational numbers are integers
  – Some exercises have answers
  – All COBOL programs have at least 20 lines
  – The sum of any two even integers is even
  – The square of any even integer is even

• Let $P(x)$ be some predicate defined for all real numbers $x$, let:
  $r = \forall x \in \mathbb{Z}, P(x);$ $s = \forall x \in \mathbb{Q}, P(x);$ $t = \forall x \in \mathbb{R}, P(x)$
  – Find $P(x)$ (but not $x \in \mathbb{Z}$) so that $r$ is true, but $s$ and $t$ are false
  – Find $P(x)$ so that both $r$ and $s$ are true, but $t$ is false
Variants of Conditionals

• Contrapositive
• Converse
• Inverse
• Generalization of relationships from before
• Examples
Necessary and Sufficient Conditions, Only If

• \( \forall x, r(x) \) is a sufficient condition for \( s(x) \) means: \( \forall x, \text{if } r(x) \text{ then } s(x) \)

• \( \forall x, r(x) \) is a necessary condition for \( s(x) \) means: \( \forall x, \text{if } s(x) \text{ then } r(x) \) (or \( \forall x, \text{if } \neg r(x) \text{ then } \neg s(x) \), if \( r(x) \) does not happen, \( s(x) \) cannot happen)

• \( \forall x, r(x) \) only if \( s(x) \) means: \( \forall x, \text{if } r(x) \text{ then } s(x) \) (or \( \forall x, \text{if } \neg s(x) \text{ then } \neg r(x) \))
Multiply Quantified Statements

- For all positive numbers $x$, there exists number $y$ such that $y < x$
- There exists number $x$ such that for all positive numbers $y$, $y < x$
- For all people $x$ there exists person $y$ such that $x$ loves $y$
- There exists person $x$ such that for all people $y$, $x$ loves $y$
- Definition of mathematical limit:
  \[ \lim_{x \to c} f(x) = L \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)(0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon) \]
- Order of quantifiers matters in some (most) cases (will find page reference, “lively discussion”):
  http://math.stackexchange.com/questions/201051/is-the-order-of-universal-existential-quantifiers-important
Negation of Multiply Quantified Statements

• The negation of $\forall x, \exists y, P(x, y)$ is logically equivalent to $\exists x, \forall y, \neg P(x, y)$

• The negation of $\exists x, \forall y, P(x, y)$ is logically equivalent to $\forall x, \exists y, \neg P(x, y)$
Prolog Programming Language

• Can use parts of logic as programming lang.
• Simple statements:
  isabove(g, b), color(g, gray)
• Quantified statements:
  if isabove(X, Y) and isabove(Y, Z) then
  isabove(X, Z)
• Questions:
  ?color(b, blue), ?isabove(X, w)
Exercises

• Determine whether a pair of quantified statements have the same truth values
  - $\forall x \in D, (P(x) \land Q(x))$ vs $(\forall x \in D, P(x)) \land (\forall x \in D, Q(x))$
  - $\exists x \in D, (P(x) \land Q(x))$ vs $(\exists x \in D, P(x)) \land (\exists x \in D, Q(x))$
  - $\forall x \in D, (P(x) \lor Q(x))$ vs $(\forall x \in D, P(x)) \lor (\forall x \in D, Q(x))$
  - $\exists x \in D, (P(x) \lor Q(x))$ vs $(\exists x \in D, P(x)) \lor (\exists x \in D, Q(x))$
Arguments with Quantified Statements

- Rule of universal instantiation: if some property is true of everything in the domain, then this property is true for any subset in the domain

- Universal Modus Ponens:
  - Premises: ($\forall x$, if $P(x)$ then $Q(x)$); $P(a)$ for some $a$
  - Conclusion: $Q(a)$

- Universal Modus Tollens:
  - Premises: ($\forall x$, if $P(x)$ then $Q(x)$); $\neg Q(a)$ for some $a$
  - Conclusion: $\neg P(a)$

- Converse and inverse errors
Validity of Arguments using Diagrams

- Premises: All human beings are mortal; Zeus is not mortal. Conclusion: Zeus is not a human being
- Premises: All human beings are mortal; Felix is mortal. Conclusion: Felix is a human being
- Premises: No polynomial functions have horizontal asymptotes; This function has a horizontal asymptote. Conclusion: This function is not a polynomial
Proof and Counterexample

• Discovery and proof

• Even and odd numbers
  – number $n$ from $\mathbb{Z}$ is called even if $\exists k \in \mathbb{Z}, n = 2k$
  – number $n$ from $\mathbb{Z}$ is called odd if $\exists k \in \mathbb{Z}, n = 2k + 1$

• Prime and composite numbers
  – number $n$ from $\mathbb{Z}$ is called prime if
    $\forall r, s \in \mathbb{Z}, n = r \times s \rightarrow r = 1 \lor s = 1$
  – number $n$ from $\mathbb{Z}$ is called composite if
    $\exists r, s \in \mathbb{Z}, n = r \times s \land r > 1 \land s > 1$
Proving Statements

• Constructive proofs for existential statements
• Example: Show that there is a prime number that can be written as a sum of two perfect squares
• Universal statements: method of exhaustion and generalized proof

• Direct Proof:
  – Express the statement in the form: $\forall x \in D, P(x) \rightarrow Q(x)$
  – Take an arbitrary $x$ from $D$ so that $P(x)$ is true
  – Show that $Q(x)$ is true based on previous axioms, theorems, $P(x)$ and rules of valid reasoning
Proof

• Show that if the sum of any two integers is even, then so is their difference

• Common mistakes in a proof
  – Arguing from example
  – Using the same symbol for different variables
  – Jumping to a conclusion
  – Begging the question
Counterexample

• To show that the statement in the form “∀x ∈ D, P(x) → Q(x)” is not true one needs to show that the negation, which has a form “∃x ∈ D, P(x) ∧ ~Q(x)” is true. x is called a counterexample.

• Famous conjectures:
  – Fermat big theorem: there are no non-zero integers x, y, z such that \(x^n + y^n = z^n\), for \(n > 2\)
  – Goldbach conjecture: any even integer can be represented as a sum of two prime numbers
  – Euler’s conjecture: no three perfect fourth powers add up to another perfect fourth power
Exercises

• Any product of four consecutive positive integers is one less than a perfect square
• To check that an integer is a prime it is sufficient to check that $n$ is not divisible by any prime less than or equal to $\sqrt{n}$
• If $p$ is a prime, is $2^p - 1$ a prime too?
• Does $15x^3 + 7x^2 - 8x - 27$ have an integer zero?