Hashing (2)
Recall: Universal Hashing

\( \mathcal{K} \) – a finite, non-empty set of hash keys

\( \mathcal{H} = \{ h_k \}_{k \in \mathcal{K}} \) – a family of hash functions

\( h_k : \mathcal{U} \to \{ 0, \ldots, m - 1 \} \), indexed by \( \mathcal{K} \)

**Def’n:** \( \mathcal{H} \) is called universal if for all \( a, b \in \mathcal{U} \) with \( a \neq b \),

\[
\left| \{ k \in \mathcal{K} : h_k(a) = h_k(b) \} \right| \leq \frac{|\mathcal{K}|}{m}.
\]

**Probabilistic interpretation:** if \( R \) is a random variable, uniformly distributed over \( \mathcal{K} \), then

\[
\Pr[h_R(a) = h_R(b)] \leq \frac{1}{m}.
\]
Constructing Universal Families

Arithmetic modulo $n$:

- $\mathbb{Z}_n$ — residue classes mod $n$:
  - $\mathbb{Z}_n = \{[0], [1], \ldots, [n - 1]\}$
  - $[a] + [b] := [(a + b) \mod n]$
  - $[a] \cdot [b] := [(a \cdot b) \mod n]$

- $\mathbb{Z}_n$ is a ring:
  - $+, \cdot$ are commutative, associative
  - $\cdot$ distributes over $+$
  - $[0]$ is the additive identity
  - $[1]$ is the multiplicative identity
Arithmetic modulo a prime $p$:

- $\mathbb{Z}_p$ is a field:
  - Every non-zero $\alpha \in \mathbb{Z}_p$ has a multiplicative inverse $\beta \in \mathbb{Z}_p$ (i.e., $\alpha \beta = 1$)
  - More generally: if $\alpha, \beta \in \mathbb{Z}_p$ with $\alpha \neq 0$, then the equation
    \[ \alpha x = \beta \]
    has a unique solution $x$
A universal family

Let $m$ be a prime, and $t$ a positive integer

Define $U := \mathbb{Z}_m^t$, $K := \mathbb{Z}_m^t$

For $k = (k_1, \ldots, k_t) \in K$, $a = (a_1, \ldots, a_t) \in U$, define

$$h_k(a) := \sum_{i=1}^{t} a_i k_i$$

Define

$$\mathcal{H} := \{h_k\}_{k \in K}$$

**Theorem:** $\mathcal{H}$ is universal
Proof of theorem.

Suppose \((a_1, \ldots, a_t) \neq (b_1, \ldots, b_t)\)

We want to count the number \(N\) of solutions \((k_1, \ldots, k_t)\) to the equation

\[
\sum_{i} a_i k_i = \sum_{i} b_i k_i.
\]

Re-write this as

\[
\sum_{i} c_i k_i = 0
\]

where \(c_i := a_i - b_i\)

By assumption, not all \(c_i\)'s are zero

Want to show: \(N \leq |\mathcal{K}|/m = m^{t-1}\)
Proof (cont’d).

Let $N$ be the number of solutions $(k_1, \ldots, k_t)$ to \( \sum_i c_i k_i = 0 \), where some $c_j \neq 0$

Want to show: $N \leq |\mathcal{K}|/m = m^{t-1}$

Without loss of generality, assume $c_1 \neq 0$

For every choice of $k_2, \ldots, k_t$, there is a unique $k_1$ such that $c_1 k_1 = -\sum_{i=2}^{t} c_i k_i$

There are $m^{t-1}$ ways of choosing $k_2, \ldots, k_t$, and each yields one solution

So $N = m^{t-1}$

QED
Practical considerations

Input space:

- View data items as bit strings of some fixed length $\ell$
- Break up into “chunks” of length $w$, where $2^w \leq m < 2^{w+1}$:
  
  \[
  \begin{array}{cccc}
  S_1 & S_2 & \cdots & S_t \\
  \end{array}
  \]

- View each $s_i$ as a number between 0 and $2^w - 1$, and each such number as an element of $\mathbb{Z}_m$
- This map from $\{0, 1\}^\ell$ to $\mathbb{Z}_m^t$ is injective
- Variable length inputs (padding)
**Practical considerations (cont’d)**

**Output space:** choice of prime \( m \)

**Bertrand’s Postulate:** There is always a prime between \( x \) and \( 2x \) for all integers \( x \geq 1 \)

**Dynamically growing the table:** when \( n/m \) gets too large, choose a new \( m' \geq 2m \), and rehash everything
Another universal family

Let $p$ be a prime, and $m$ a positive integer

Define $\mathcal{U} := \{0, \ldots, p - 1\}$,
\[ \mathcal{K} := \{1, \ldots, p - 1\} \times \{0, \ldots, p - 1\} \]

For $k = (k_1, k_2) \in \mathcal{K}$, $a \in \mathcal{U}$, define
\[ h_k(a) := \left( (k_1a + k_2) \mod p \right) \mod m \]

**Theorem:** $\mathcal{H} := \{h_k\}_{k \in \mathcal{K}}$ is universal (see text)

**Pros:** free choice of $m$

**Cons:** multiplication of large numbers
General problem: large key space — almost as large as the input space

Solution: weaker (but still useful) hashing requirements
ε-universal Hashing

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\( \mathcal{H} = \{ h_k \}_{k \in \mathcal{K}} \) – a family of hash functions

\( h_k : \mathcal{U} \rightarrow \{0, \ldots, m - 1\} \), indexed by \( \mathcal{K} \)

**Def’n:** Let \( 0 \leq \varepsilon \leq 1 \). \( \mathcal{H} \) is called \( \varepsilon \)-universal if for all \( a, b \in \mathcal{U} \) with \( a \neq b \),

\[
\left| \{ k \in \mathcal{K} : h_k(a) = h_k(b) \} \right| \leq \varepsilon \cdot |\mathcal{K}|.
\]

**Probabilistic interpretation:** if \( R \) is a random variable, uniformly distributed over \( \mathcal{K} \), then

\[
\Pr[h_R(a) = h_R(b)] \leq \varepsilon
\]

universal = \((1/m)\)-universal
Using $\epsilon$-universal hash families

As long as $\epsilon$ is not too big, many of the results we proved have useful analogs.

E.g., in a table with at most $n$ data items

- expected cost of each dictionary operation in a table containing $n$ items is $\leq 1 + \epsilon n$.
- expected value of maximum load is $\leq \sqrt{\epsilon n^2 + n}$

Building $\epsilon$-universal hash families

We can get by with much shorter hash keys
More about modular arithmetic

Let \( p \) be a prime

Let \( f = c_d X^d + c_{d-1} X^{d-1} + \cdots + c_1 X + c_0 \) be a polynomial with coefficients in \( \mathbb{Z}_p \), with \( c_d \neq 0 \) (the degree of \( f \) is \( d \))

We can evaluate \( f \) at a point \( u \in \mathbb{Z}_p \): \( f(u) = \sum_i c_i u^i \in \mathbb{Z}_p \)

Then \( f \) has at most \( d \) roots in \( \mathbb{Z}_p \):

\[
\left| \{ u \in \mathbb{Z}_p : f(u) = 0 \} \right| \leq d.
\]

This is a general fact that holds for any field (e.g., the reals, the complex numbers, \( \mathbb{Z}_p \)), but not for arbitrary rings (e.g., \( \mathbb{Z}_n \)).
An $\varepsilon$-universal family

Let $m$ be a prime, and $t$ a positive integer

Define $\mathcal{U} := \mathbb{Z}_m^{t+1}$, $\mathcal{K} := \mathbb{Z}_m$

For $k \in \mathcal{K}$, $a = (a_0, a_1, \ldots, a_t) \in \mathcal{U}$, define

$$h_k(a) := \sum_{i=0}^{t} a_ik^i$$

Define

$$\mathcal{H} := \{h_k\}_{k \in \mathcal{K}}$$

**Theorem:** $\mathcal{H}$ is $(t/m)$-universal
Proof. Suppose \((a_0, \ldots, a_t) \neq (b_0, \ldots, b_t)\)

We want to count the number \(N\) of solutions \(k\) to the equation

\[
\sum_i a_i k^i = \sum_i b_i k^i.
\]

Re-write this as

\[
\sum_i c_i k^i = 0
\]

where \(c_i := a_i - b_i\)

\(N = \#\) roots of \(\sum_i c_i X^i\), which is a non-zero polynomial of degree at most \(t\)

\[
\therefore N \leq t = (t/m) \cdot m. \quad \text{QED}
\]
Pairwise Independent Hashing: a stronger notion

\( \mathcal{K} \) – a finite, non-empty set of hash keys

\( \mathcal{H} = \{ h_k \}_{k \in \mathcal{K}} \) – a family of hash functions

\( h_k : \mathcal{U} \to \{ 0, \ldots, m - 1 \} \), indexed by \( \mathcal{K} \)

**Def’n:** \( \mathcal{H} \) is called pairwise independent if for all \( a, b \in \mathcal{U} \) with \( a \neq b \), and for all \( r, s \in \{ 0, \ldots, m - 1 \} \), we have

\[
\left| \{ k \in \mathcal{K} : h_k(a) = r \text{ and } h_k(b) = s \} \right| = \frac{|\mathcal{K}|}{m^2}.
\]
Probabilistic interpretation

Let $R$ be a random variable, uniformly distributed over $\mathcal{K}$

For each $a \in \mathcal{U}$, define the random variable $V_a := h_R(a)$

**Claim:** if $|\mathcal{U}| > 1$, then each $V_a$ is uniformly distributed over $\{0, \ldots, m-1\}$

**Proof.** Let $a \in \mathcal{U}$, $r \in \{0, \ldots, m-1\}$

Let $b \in \mathcal{U}$, $b \neq a$. Then:

\[
\Pr[V_a = r] = \sum_s \Pr[V_a = r \text{ and } V_b = s] = \sum_s 1/m^2 = 1/m.
\]
Recall: $X$ and $Y$ are **independent** if for all possible $x$ and $y$,

$$\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \Pr[Y = y],$$

or equivalently

$$\Pr[X = x] = \Pr[X = x \mid Y = y].$$

A family of random variables $\{X_i\}_{i \in \mathcal{I}}$ is called **pairwise independent** if $X_i$ and $X_j$ are independent for all $i \neq j$. 
The family of random variables \( \{ V_a \}_{a \in \mathcal{U}} \) is pairwise independent:

\[
Pr[V_a = r \text{ and } V_b = s] = \frac{1}{m^2} = Pr[V_a = r] \cdot Pr[V_b = s]
\]

pairwise independent \( \Rightarrow \) universal:

\[
Pr[V_a = V_b] = \sum_s Pr[V_a = s \text{ and } V_b = s]
\]

\[
= \sum_s \frac{1}{m^2} = \frac{1}{m}
\]
A pairwise independent family

Let $m$ be a prime, and $t$ a positive integer

Define $\mathcal{U} := \mathbb{Z}^t_m$, $\mathcal{K} := \mathbb{Z}^{t+1}_m$

For $k = (k_0, k_1, \ldots, k_t) \in \mathcal{K}$, $a = (a_1, \ldots, a_t) \in \mathcal{U}$, define

$$h_k(a) := k_0 + \sum_{i=1}^t a_i k_i$$

Define

$$\mathcal{H} := \{h_k\}_{k \in \mathcal{K}}$$

**Theorem:** $\mathcal{H}$ is pairwise independent (Homework)
Application: message authentication

Alice and Bob share a random key $R$

Later, Alice sends a message $a$ to Bob, together with a hash code $C := h_R(a)$

An adversary can try to fool Bob, by sending him a different message with a correct hash code, that is, a message $B$, and a hash code $D$, such that $B \neq a$ and $h_R(B) = D$

Here, $B$ and $D$ are functions of $C$

(Homework) Pairwise independent hashing implies

$$\Pr[B \neq a \text{ and } h_R(B) = D] \leq \frac{1}{m}$$