All pairs shortest paths

One approach:

- Run a single-source shortest path algorithm from each vertex
  - Bellman-Ford (no negative cycles): \( O(|V|^2(|V| + |E|)) \), or \( O(|V|^4) \)
  - Dijkstra (no negative edges): \( O(|V|(|V| \log |V| + |E|)) \), or \( O(|V|^3) \)

Floyd-Warshall Algorithm:

- no negative cycles
- running time \( O(|V|^3) \)
• Number the vertices 1..n

• For a path \( p = \langle v_0, v_1, \ldots, v_{k-1}, v_k \rangle \), we say that \( v_1, \ldots, v_{k-1} \) are intermediate vertices

• For \( k = 0 \ldots n \), \( \delta^{(k)}(i, j) \) := length of the shortest path from \( i \) to \( j \) whose intermediate vertices belong to \( \{1, \ldots, k\} \)

\[
\delta^{(0)}(i, j) = \begin{cases} 
0 & \text{if } i = j; \\
w(i, j) & \text{if } i \neq j \text{ and } (i, j) \in E \\
\infty & \text{otherwise}
\end{cases}
\]

• For \( k > 0 \)

\[
\delta^{(k)}(i, j) = \min \left( \delta^{(k-1)}(i, j), \delta^{(k-1)}(i, k) + \delta^{(k-1)}(k, j) \right)
\]
Straightforward implementation:

- Use a 3D array $D[i, j, k]$

$$
D[i, j, 0] \leftarrow \delta^{(0)}(i, j) \text{ for } i, j = 1 \ldots n
$$

for $k \leftarrow 1 \text{ to } n$

for $i \leftarrow 1 \text{ to } n$

for $j \leftarrow 1 \text{ to } n$

    $d' \leftarrow D[i, k, k - 1] + D[k, j, k - 1]$

    if $d' < D[i, j, k - 1]$
        then $D[i, j, k] \leftarrow d'$
    else $D[i, j, k] \leftarrow D[i, j, k - 1]$

- Running time: $O(n^3)$
- Space: $O(n^3)$
Improving the space requirement:

- Since $D[\cdot, \cdot, k]$ depends only on $D[\cdot, \cdot, k-1]$, we can obviously get by with just two 2D arrays.
- In fact, we can get by with just a single array, with updates “in place”.

Justification:

- $\delta^{(k)}(i, k) = \delta^{(k-1)}(i, k)$
- $\delta^{(k)}(k, j) = \delta^{(k-1)}(k, j)$
- Why? No negative cycles.
- So in the formula:
  $$\delta^{(k)}(i, j) = \min(\delta^{(k-1)}(i, j), \delta^{(k-1)}(i, k) + \delta^{(k-1)}(k, j))$$
  
  these don’t change in loop iteration $k$.
Improved implementation:

- Use a 2D array $D[i, j]$

$$D[i, j] \leftarrow \delta^{(0)}(i, j) \text{ for } i, j = 1 \ldots n$$

for $k \leftarrow 1$ to $n$ do

  for $i \leftarrow 1$ to $n$ do

    for $j \leftarrow 1$ to $n$ do

      $d' \leftarrow D[i, k] + D[k, j]$

      if $d' < D[i, j]$

        then $D[i, j] \leftarrow d'$
Adding path recovery:

- Two arrays: $D[i,j]$, $N[i,j]$
- $N[i,j] =$ next vertex in the shortest path from $i$ to $j$

\[
D[i,j] \leftarrow \delta^{(0)}(i,j) \text{ for } i, j = 1 \ldots n
\]
\[
N[i,j] \leftarrow j \text{ for } i, j = 1 \ldots n
\]

for $k \leftarrow 1$ to $n$ do
  for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
      \[
d' \leftarrow D[i,k] + D[k,j]
      \]
      if $d' < D[i,j]$
      then
        \[
        D[i,j] \leftarrow d'
        \]
        \[
        N[i,j] \leftarrow N[i,k]
        \]

Printing a shortest path from $u$ to $v$:

\[
x \leftarrow u, \text{ print } x
\]
\[
\text{while } x \neq v \text{ do: } x \leftarrow N[x, v], \text{ print } x
\]
Johnson’s Algorithm

Motivation:

• For sparse graphs with no negative edges, running Dijkstra from each node is faster than Floyd-Warshall

• But what if we have a sparse graph with negative edges, and no negative cycles?

Idea:

• Convert $G$ to a graph $G'$ such that
  ○ $G'$ has no negative edges
  ○ Shortest paths in $G'$ are shortest paths in $G$

• How? Re-weighting
Let $G = (V, E)$ be a directed graph with weights $w : E \to \mathbb{R}$, and no negative cycles.

Let $h : V \to \mathbb{R}$, and define
$$\tilde{w}(u, v) = w(u, v) + h(u) - h(v)$$

For a path $p$ in $G$, define $\tilde{w}(p)$ to be the weight of the path, using $\tilde{w}$.

Observation: if $p$ is a path from $u$ to $v$, then
$$\tilde{w}(p) = w(p) + h(u) - h(v) \quad \text{(telescoping sum)}$$

No negative cycles relative to $\tilde{w}$.

Define $\hat{\delta}(u, v)$ to be the length of the shortest path from $u$ to $v$, relative to $\tilde{w}$.

Therefore, any shortest path relative to $\tilde{w}$ is also a shortest path relative to $w$, and
$$\hat{\delta}(u, v) = \delta(u, v) + h(u) - h(v)$$
Johnson’s Algorithm:

- Construct a new weighted graph $G'$ by adding a node $s$ to $G$ and 0-weight edges connecting $s$ to every other vertex in $G$
- Run Bellman-Ford starting from $s$, to obtain $\delta(s, v)$ for all nodes $v$
- Define $h(v) := \delta(s, v)$, and re-weight as above
- By Triangle Inequality, for every edge $(u, v)$:
  \[
  h(v) = \delta(s, v) \leq \delta(s, u) + w(u, v) = h(u) + w(u, v)
  \]
  and so
  \[
  \hat{w}(u, v) = w(u, v) + h(u) - h(v) \geq 0
  \]
- Run Dijkstra from every node, using $\hat{w}$
Running time:

- Bellman-Ford: $O(|V|(|V| + |E|))$
- Dijkstra: $O(|V|(|V| \log |V| + |E|))$
- Dijkstra dominates