CSCI-UA.0480-004
Algorithmic Problem Solving
Lecture 17

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Spring 2015
• Homework 7 due tomorrow (Friday) **at 5pm**
• Homework 8 will be posted around the same time
• Final Exam:
  • Thursday, May 14 from 4pm to 5:50pm in WWH 101
  • Must bring a Laptop! (Email me *asap* if you do not have one!)
• Google Code Jam Qualification Round starts tomorrow!
  • Not too late to register
  • +2 bonus points if you pass qualification round
  • +4 bonus points if you pass online round #1
A random variable $X$ maps outcomes of an experiment to a real number.

- E.g., let $X$ be the number of heads in 3 coin flips, so $X \in \{0, 1, 2, 3\}$

- Notation: $P(X=2)$: probability that the number of heads in 3 coin flips is 2

- A random variable is **discrete** if its set of possible values is finite

- The **distribution** of $X$ is the distribution of probabilities of all possible values of $X$
• For a discrete random variable $X$, **probability mass function** $f$ is $f(x) = P(X = x)$
  • E.g., $T =$ sum of two dice rolls
    • $f(2) = 1/36$, $f(3) = 2/36$, etc…
  • The **support** of the distribution of $X$ is the set of all values $x$ such that $f(x) > 0$
• The **cumulative distribution function** $F_X$ of $X$ is $F_X(x) = P(X \leq x)$
  • CDFs are always non-decreasing
• Bernoulli: coin flip; heads w/ probability p; X~Bern(p)
  • PMF: P(X=0) = 1-p, P(X=1) = p
  • CDF: F(0) = 1-p, F(1) = 1
  • E[X] = p

• Binomial: flip n coins, get k heads; X~Bin(n, p)
  • PMF: P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}
  • E[X] = np

• Geometric: number of coin flips before getting heads; X~Geom(p)
  • PMF: P(X=k) = (1-p)^{k-1}p
  • CDF: F(k) = 1-(1-p)^k
  • E[X] = 1/p

• Others: Poisson, Weibull, hypergeometric, …
• A random variable $X$ is **continuous** if its CDF $F$ is differentiable
• The **probability density function** is simply $f(x) = F'(x)$

$$F(x) = \int_{-\infty}^{x} f(t) dt$$

• How to calculate probabilities for a continuous variable?

$$P(a \leq X < b) = F(b) - F(a) = \int_{a}^{b} f(t) dt$$
Exercise: What is the probability of getting dealt a full house (standard 52-card deck; full house means a 3-of-a-kind and a 2-of-a-kind)?
• Take a probability / stats class if you haven’t already!
  • “I keep saying the sexy job in the next ten years will be statisticians… The ability to take data - to be able to understand it, to process it, to extract value from it, to visualize it, to communicate it's going to be a hugely important skill in the next decades…”
    --- Hal Varian, Chief Economist for Google, 2009

• Harvard’s Stats 110 class
  • Joseph Blitzstein’s lectures are on YouTube/iTunesU and handouts/assignments/exams (with solutions) are online
  • His assignments are interesting!
• Java and C++ each have a function that will give you a pseudo-random number from the uniform distribution [0, 1]
• But how do you sample from an arbitrary distribution?
• tl;dr:
  • Let U be a standard uniform random variable
  • Let X be any continuous random variable whose CDF F is strictly increasing over its support
    • Strictly increasing so that $F^{-1}$ is well defined
  • Then $F^{-1}(U) \sim X$
• How generate numbers sampled from Expo(1)?
  • PDF: $p(x) = e^{-x}$
  • CDF: $F(x) = 1-e^{-x}$
• $F(x) = 1-e^{-x} \Rightarrow F^{-1}(x) = -\log(1-x)$
• To generate random numbers from Expo(1) distribution:
  • Generate $x \sim \text{Unif}[0, 1]$, then $-\log(1-x) \sim \text{Expo}(1)$
• C++11 offers the <random> header
  • Ability to transform uniform random numbers into other common distributions, e.g., Bernoulli, binomial, geometric, etc.
Theorem 1: Let $X$ be any continuous random variable whose CDF $F$ is strictly increasing over its support. Then $F(X) \sim U$.

Proof: Let $Y = F(X)$. Goal: show that $F_Y(u) = u$ for $u \in [0, 1]$ (the CDF of the uniform):

$$F_Y(u) = P(Y \leq u)$$
$$= P(F(X) \leq u)$$
$$= P(X \leq F^{-1}(u))$$
$$= F(F^{-1}(u))$$
$$= u$$
**Theorem 2:** Let $X$ be any continuous random variable whose CDF $F$ is strictly increasing over its support. Then $F^{-1}(U) \sim X$.

**Proof:** Let $Y = F^{-1}(U)$. Goal: show $F_Y(u) = F(u)$:

\[
F_Y(u) = P(Y \leq u) \\
= P(F^{-1}(U) \leq u) \\
= P(U \leq F(u)) \\
= F(u)
\]
Exercise: Write a function that will return a point in a unit circle (disk) such that each point is returned with equal probability (you may use a function rand() which returns a float from [0, 1] with uniform probability)
• Pick a random $r$, $\theta$
  • $\theta$ can be chosen uniformly, but…
  • How do you pick $r$ such that the points distribute uniformly along it?

• What is the PDF for picking a point along the radius?
  • Consider a slice of the disk
  • Note that $P(X=r) \propto r$
  • How to translate that into a PDF?
Exercise: Given a fair coin, how can you simulate a 6 sided die? What is the expected number of flips to do this?
Exercise: You are given an n-sided die where side $i$ has probability $p_i$ of being rolled. What is the most efficient data structure for simulating rolls of the die?