Exercise 7.1. Consider the following table of 5 points \( x_0, \ldots, x_4 \) (which are not equally spaced) and corresponding values of two different functions, \( f \) and \( g \), evaluated at \( \{x_i\} \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f_i )</th>
<th>( g_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3.1</td>
<td>3.1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>2.8</td>
<td>3.8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>2.0</td>
<td>4.1</td>
</tr>
</tbody>
</table>

(a) Interpolate the values of \( \{f_i\} \) using

(i) the interpolating polynomial of degree at most 4 (using, for example, Matlab’s `polyfit` and `polyval` commands);

(ii) the ‘not a knot’ default spline computed by Matlab’s `spline` command;

(iii) the ‘clamped’ spline computed by Matlab in which first derivatives at the endpoints of the interval are forced to be zero, as illustrated in the second example shown in Matlab’s ‘help’ about `spline`;

(iv) the shape-preserving piecewise cubic computed by Matlab’s `pchip` command.

Using these built-in commands, calculate the values of the interpolating polynomial, the splines, and the pchip interpolants at a reasonable number (say, 41) of equally spaced points in the interval defined by the points \( \{x_i\} \). Plot these interpolants both separately and superimposed.

Comment on how the interpolated functions differ qualitatively. Which choice of interpolant seems to provide the best “feeling” for the data? Explain why. Are any of the interpolants unsatisfactory (in your opinion)? Explain.

(b) Repeat part (a), this time interpolating the values of \( \{g_i\} \). Be sure to comment on qualitative differences (if any) between the interpolants, and on any qualitative differences from the results of part (a).

(c) Make up your own set of function values to be associated with the given 5 points \( \{x_i\} \), choosing the function values so that “interesting” things happen. Comment on how the various interpolants differ qualitatively for your data. Which choice of interpolant seems to you to provide the best feeling for your data? Explain why.

Exercise 7.2. Suppose that you are given the following tabulated data \( \{x_i, f_i\}, i = 0, \ldots, 3 \), in the interval \([0, 4]\), where \( f_i = f(x_i) \) for some function \( f \):

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.52</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.68</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2.12</td>
</tr>
</tbody>
</table>
Assume that interval $i$ means $[x_{i-1}, x_i]$, and note that the values $\{x_i\}$ are not equally spaced.

(a) Suppose that we want to obtain the coefficients for the three cubic functions that define the “natural” spline interpolant for these data, where a “natural” spline has the property that its second derivative must be zero at $a$ and $b$.

As described on page 320 of Lecture 17, the coefficients of the three cubics can be obtained by solving a $9 \times 9$ linear system. Compute and display the associated matrix $A$ and right-hand side $v$; then solve the linear system and give the coefficients for the three cubics.

(b) Suppose that, instead, we want a “clamped” spline, in which first derivatives of the spline are specified at the two endpoints of the interval. (See page 318 of Lecture 17.) How would this change the elements of the matrix $A$ and/or the right-hand side $v$?

**Exercise 7.3.** Given two finite intervals $[a, b]$ and $[\alpha, \beta]$, we can map $x \in [\alpha, \beta]$ to $t \in [a, b]$ by a linear transformation of the form $t = cx + d$. What are the coefficients $c$ and $d$, and what is the associated expression for $t$ in terms of $x$, $a$, $b$, $\alpha$, and $\beta$?

**Exercise 7.4.** Given a scalar $b > 1$, let $I(b)$ denote the exact integral

$$I(b) = \int_1^b \frac{dt}{t} = \ln b.$$

For a given value of $b$, let $M(b)$ denote the estimate of $I(b)$ from the midpoint rule; $T(b)$ the estimate of $I(b)$ from the trapezoid rule; and $S(b)$ the estimate of $I(b)$ from Simpson’s rule. For $b = 1.5$ and $b = 2$, do the following:

(i) compute the exact integral $I(b)$;
(ii) compute $M(b)$ and the error $I(b) - M(b)$;
(iii) compute $T(b)$ and the error $I(b) - T(b)$;
(iv) compute $S(b)$ and the error $I(b) - S(b)$;
(v) comment on how well the actual errors correspond to the estimates derived in class. Why are the error estimates more accurate for the first value of $b$?

**Exercise 7.5**

(a) A rule of thumb in numerical quadrature is that the magnitude of the error from using the trapezoid rule is about double the magnitude of the error from using the midpoint rule. Consider the integral

$$\int_{-1}^{1} x^4 dx.$$

Give the results of applying (i) the midpoint rule and (ii) the trapezoid rule to estimate this integral, along with the error in each case. If the rule of thumb does not apply, explain why.

(b) Explain why Simpson’s rule is exact for the integral

$$\int_{-1}^{1} x^2 dx.$$

**Exercise 7.6.** Consider the integral (and its known exact value)

$$\int_{0}^{\pi} \sin^2 4t \, dt = \frac{\pi}{2}.$$

Explain what would go wrong if you tried to approximate this integral with the midpoint rule, the trapezoid rule, or Simpson’s rule. How could you obtain an accurate approximation to this integral?