Numerical Computing
CSCI-UA.0421-001
New York University, Spring 2015
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Homework Assignment 4
Assigned February 26, 2015; due March 5, 2015

In all problems involving vector or matrix norms, assume unless stated otherwise that the relevant two-norm is used. When equations involving matrices and vectors are given, assume that the dimensions of the matrices and vectors are compatible.

Exercise 4.1.
(a) Show that, if the square matrices $B$ and $C$ are nonsingular, then $\text{cond}(BC) \leq \text{cond}(B) \text{cond}(C)$, where the condition number is measured in any of the “standard” matrix norms (one, two, or infinity).
(b) Show that $\text{cond}(A^T) = \text{cond}(A)$, measured in the matrix two-norm.
(c) If cond is measured in the matrix one-norm (the maximum absolute column sum), is $\text{cond}(A^T) = \text{cond}(A)$? Explain your answer, giving a specific example for illustration if your answer is “no”.

Exercise 4.2. Given a nonsingular matrix $A$ and a vector $b$, let $x$ denote the exact solution of $Ax = b$.
(a) If $\|b\| = 10^{-6}$ and $\|x\| = 1$, explain what, if anything, can be deduced about the smallest singular value of $A$?
(b) If the conditions of part (a) apply and $\|A\| = 1$, what do we know about $\text{cond}(A)$?
(c) If $\|A\| = 1$, $\|b\| = 1$ and $\|x\| = 1$, is $A$ guaranteed to be well conditioned? Explain why or why not, preferably giving an illustrative example.

Exercise 4.3. Consider the matrix $A$ and vector $b$,

$$A = \begin{pmatrix} 0.671 & -0.273 \\ -0.335 & 0.136 \end{pmatrix}, \quad b = \begin{pmatrix} 0.398 \\ -0.199 \end{pmatrix},$$

and let $x$ be the exact solution of $Ax = b$.

(a) Give the singular value decomposition of $A$. What is $\text{cond}(A)$?
(b) Give a specific small perturbation vector $\delta_1$ such that the solution $x_1$ of $Ax_1 = b + \delta_1$ satisfies

$$\frac{\|x - x_1\|}{\|x\|} \approx \text{cond}(A) \frac{\|\delta_1\|}{\|b\|}.\quad(4.2)$$

Include the numerical values of $\delta_1$, $\|\delta_1\|$, $x_1$, $\|x - x_1\|$, the ratios $\|x - x_1\|/\|x\|$ and $\|\delta_1\|/\|b\|$, and the value on the right-hand side of (4.2). Explain how you chose $\delta_1$, referring to the SVD of $A$.
(c) Give a specific small perturbation vector $\delta_2$ such that the solution $x_2$ of $Ax_2 = b + \delta_2$ satisfies

$$\frac{\|x - x_2\|}{\|x\|} \ll \text{cond}(A) \frac{\|\delta_2\|}{\|b\|}.\quad(4.3)$$

where $\ll$ means “is much less than”. Please give $\delta_2$, $\|\delta_2\|$, $x_2$, $\|x - x_2\|$, the ratios $\|x - x_2\|/\|x\|$ and $\|\delta_2\|/\|b\|$, and the value on the right-hand side of (4.3). Explain how you chose $\delta_2$, referring to the SVD.
(d) Choose two specific small “random” perturbation vectors \( \delta_i \), for \( i = 3 \) and \( i = 4 \). Compute the solutions \( x_3 \) and \( x_4 \) of the associated linear systems \( Ax_i = b + \delta_i \), for \( i = 3 \) and \( i = 4 \). Show the ratios \( \|x - x_i\|/\|x\| \), \( \|\delta_i\|/\|b\| \), and the value \( \text{cond}(A)\|\delta_i\|/\|b\| \) for \( i = 3 \) and \( i = 4 \). Comment on and explain the results, with particular attention to any resemblance (or lack of resemblance) to the results of (b) and (c).

(e) Let \( b_A = (-45, 22)^T \). Compute the solution of \( Ax = b_A \) using Matlab’s backslash operation. Devise your own perturbation matrix \( \delta A \) that is small in norm such that the solution \( \tilde{x}_A \) of \( (A + \delta A)\tilde{x}_A = b_A \) satisfies

\[
\frac{\|x - \tilde{x}_A\|}{\|x\|} \approx \text{cond}(A) \frac{\|\delta A\|}{\|A\|}.
\]

Please give \( \delta A \), \( \|\delta A\| \), \( \tilde{x}_A \), \( \|x - \tilde{x}_A\|/\|x\| \), the ratios \( \|x - \tilde{x}_A\|/\|x\| \) and \( \|\delta A\|/\|A\| \), and the value of \( \text{cond}(A)\|\delta A\|/\|A\| \). Explain how you chose \( \delta A \).

**Exercise 4.4.** Suppose that \( R \) is an \( n \times n \) upper-triangular matrix and that \( k \) of its diagonal elements are zero, where \( n > k > 0 \). We know that \( R \) is singular, but is the rank of \( R \) necessarily \( n - k \)? If “yes”, show why. If “no”, give an example that shows why not.

**Exercise 4.5.**

(a) Show that the product of two square upper-triangular matrices is upper triangular.

(b) Given a nonsingular upper-triangular matrix \( U \) whose diagonal elements are \( \{u_{ii}\} \), show that (i) its inverse \( U^{-1} \) is also upper triangular and (ii) the diagonal elements of \( U^{-1} \) are the reciprocals of the diagonal elements of \( U \).

(c) Using the result of (b) (and recalling that the matrix infinity norm is the maximum absolute row sum), show that

\[
\|U\|_{\infty} \geq \max_i |u_{ii}| \quad \text{and} \quad \|U^{-1}\|_{\infty} \geq \frac{1}{\min_i |u_{ii}|}.
\]

These two inequalities imply that, measured in the infinity norm,

\[
\text{cond}(U)_{\infty} \geq \frac{\max_i |u_{ii}|}{\min_i |u_{ii}|}.
\] (4.4)

In practice, the number on the right-hand side is often used as an estimate of the condition of an upper-triangular matrix.

**Exercise 4.6.** Consider the matrix

\[
A = \begin{pmatrix}
0.932165 & 0.443126 & 0.417632 \\
0.712345 & 0.915312 & 0.887652 \\
0.632165 & 0.514217 & 0.493909
\end{pmatrix}.
\]

(a) Compute the \( LU \) decomposition of \( A \). What feature of \( U \) shows immediately that \( A \) is ill-conditioned?

(b) Compute \( \text{cond}(A) \), \( \text{cond}(U) \), and \( \text{cond}(L) \), measured in the matrix infinity-norm.

(c) Compute the ratio on the right-hand side of (4.4). Is the value of that ratio a good estimate of \( \text{cond}(U) \) and \( \text{cond}(A) \), measured in the infinity norm, in this case?

**Exercise 4.7.** Let \( A \) be a nonsingular \( n \times n \) matrix with smallest singular value \( \sigma_n > 0 \).

(a) Let \( B \) be any \( n \times n \) singular matrix. Show that \( \|A - B\|_2 \geq \sigma_n \).
(b) Explain how to use the SVD of $A$ to construct a singular matrix $B$ such that $\|A - B\| = \sigma_n$, where $\|\cdot\|$ is the two-norm. Give a mathematical form for the resulting $B$, expressed in terms of quantities from the SVD of $A$.

(c) Is $B$ necessarily unique? Explain why or why not.

(d) Given the matrix $A$ from (4.1),

$$A = \begin{pmatrix} 0.671 & -0.273 \\ -0.335 & 0.136 \end{pmatrix},$$

use the technique from part (b) of this problem to compute a matrix $B$ such that $B$ is singular and $\|B - A\| = \sigma_n$, showing at least 6 figures for each entry of $B$. Compute the SVD of $B$, give its computed singular values, and give the computed value of $\|A - B\|$. If the smallest computed singular value of $B$ is not exactly zero, comment on why this might be the case.