Problem 2-1 (Counting Inversions, Part 1) 10 points

Let $A[1, \ldots, n]$ be an array of $n$ distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair $(i, j)$ is called an inversion of $A$.

(a) (2 points) List the five inversions of the array $⟨2, 3, 8, 6, 1⟩$.

(b) (3 points) Which arrays with elements from the set $\{1, 2, \ldots, n\}$ have the smallest and the largest number of inversions and why? State the expressions exactly in terms of $n$.

(c) (5 points) What is the relationship between the running time of INSERTION_SORT and the number of inversions $I$ in the input array? Justify your answer.

Problem 2-2 (Counting Inversions, Part 2) 16 points

Let $A[1 \ldots n]$ be an array of pairwise different numbers. We call pair of indices $1 \leq i < j \leq n$ an inversion of $A$ if $A[i] > A[j]$. The goal of this problem is to develop a divide-and-conquer based algorithm running in time $Θ(n \log n)$ for computing the number of inversions in $A$.

(a) (8 points) Suppose you are given a pair of sorted integer arrays $A$ and $B$ of length $n/2$ each. Let $C$ an $n$-element array consisting of the concatenation of $A$ followed by $B$. Give an algorithm (in pseudocode) for counting the number of inversions in $C$ and analyze its runtime. Make sure you also argue (in English) why your algorithm is correct.

(b) (8 points) Give an algorithm (in pseudocode) for counting the number of inversions in an $n$ element array $A$ that runs in time $Θ(n \log n)$. Make sure you formally prove that your algorithm runs in time $Θ(n \log n)$ (e.g., write the recurrence and solve it.)

(Hint: Combine Merge Sort with part (a).)

Problem 2-3 (The Same Outcome in Different Ways) 15 Points

Consider the following recurrence $T(n) = 4T(n/2) + n^2 \log n$, $T(1) = 1$.

(a) (2 points) Can the master’s theorem, as stated in the book, be applied to solve this recurrence? If yes, apply it. If not, formally explain the reason why.

(b) (4 points) Solve the above recurrence using the recursion tree method.

(c) (4 points) Formally verify that your answer from part (b) is correct using induction.

(d) (5 points) Solve the above recurrence exactly using domain-range substitution.
Problem 2-4 (Tricky Recurrence) 5 points

Solve precisely the recurrence $T(n) = T(\sqrt{n}) + \log n$, with $T(2) = 2$ (assume $n$ is such that $\sqrt{\cdots \sqrt{n}}$ is always an integer). (Hint: Substitute $n = 2^k$... Then substitute again.)