The following assignment is due on March 12 at 5:00 pm. Turn in a listing of your MATLAB programs as well as output from selected runs as well as a discussion. You can instead use any other standard programming language. Please print everything rather than sending it by e-mail.

This is a series of exercises related to polynomial interpolation on the interval $[-1, +1]$ interpolating at the Chebyshev points $\cos(j\pi/n)$ $j = 0, \ldots, n$. Instead of the standard monomial basis for the space of polynomials, we can use the Chebyshev polynomials $T_k(x)$, $k = 0, \ldots, n$.

1. Show that the coefficients of the Chebyshev series which interpolates given values $f_i$, $i = 0, \ldots, n$, can be computed by using FFT, in particular a cosine transform. Write a program that implements this algorithm. Note that the Chebyshev series expansion will have rapidly decaying coefficients if the function we interpolate is smooth; the same is true of Fourier series expansions of periodic functions.

2. The Clenshaw-Curtis numerical quadrature rule is derived by integrating the Chebyshev series that interpolates a given function at the Chebyshev nodes and using the fact that $\int_{-1}^{+1} T_k(x) dx = \frac{2}{2^k} \pi$ for $k = 0, 2, 4, \ldots$, while the integrals of $T_k$ vanishes for all odd $k$. Use convenient values of $n$ and study the rate of convergence when $n$ successively is doubled for some nice, smooth functions as well as for the rather nasty function

$$e^{x^2}[sech(4\sin(40x))]^{\exp(x)}.$$ 

Recall that $sech(x) := \cosh(x)^{-1}$. This function is borrowed from formula (19.11) in Trefethen’s recent book on Approximation Theory and Practice.

3. Develop a spectral method to solve boundary value problems for ordinary differential equations of second order, defined on the interval $[-1, +1]$ and with Dirichlet conditions at the end points of the interval. The approximate solution is defined by a polynomial of degree $n$ and the unknowns
are simply the values at the interior Chebyshev points. The approximation of first and second derivatives are obtained by collocation, i.e., simply by term by term differentiation. In the handout on barycentric interpolation, you should have a look at formula (5.8) from which relatively simple expression of the derivatives can be obtained. Show that for the first derivative, we obtain the elements of the matrix, that represents the first derivative, by

\[ D_{ij} = \ell'(x_i) = \frac{\lambda_j}{\lambda_i(x_i - x_j)}, \quad i \neq j, \]

while

\[ D_{ii} = \frac{x_i}{1 - x_i^2}. \]

Also develop formulas for the second derivative.

For each ODE, a linear system of equations results with one equation for each Chebyshev point in the interior of the interval and two equations which enforce the Dirichlet conditions at the endpoints.

Try your program for two problems borrowed from chapter 21 of Trefethen’s book:

\[ u'' + u' + 100u = x, \quad u(-1) = u(1) = 0, \]

and

\[ u'' - xu = 0, \quad u(-30) = 1, \quad u(30) = 0. \]

In the second case, we need to map the problem to the interval \([-1, +1]\).

It might be of interest to interpolate the resulting approximate solutions using Chebyshev polynomials. The decay of the coefficients, when an adequate number of points have been used, should give an indication that we have obtained an accurate solution for the ODE.