1. Let $L = L(\mathbf{b}_1, \ldots, \mathbf{b}_n) \subseteq \mathbb{R}^n$ be a rank $n$ lattice and let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be the Gram-Schmidt orthogonalization of $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

(a) Show that it is not true in general that $\lambda_n(L) \geq \max_i \|\mathbf{b}_i\|_2$.

(b) Show that for any $j = 1, \ldots, n$, $\lambda_j(L) \geq \min_{i=1,\ldots,n} \|\mathbf{b}_i\|_2$.

(c) Show that for any $x \in \mathbb{R}^n$, there exists a point $y \in L$ such that $\|x - y\|_2^2 \leq \frac{1}{4} \sum_{i=1}^n \|\mathbf{b}_i\|_2^2$.

(d) Show that $C = \{x \in \mathbb{R}^n : -\frac{1}{2}\|\mathbf{b}_i\|_2^2 \leq \langle x, \mathbf{b}_i \rangle < \frac{1}{2}\|\mathbf{b}_i\|_2^2, \forall i \in [n] \}$ is a fundamental domain of $L$.

(e) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in L$ be linearly independent vectors. Show that there exists a basis $\mathbf{y}_1, \ldots, \mathbf{y}_n$ of $L$ such that $\|\mathbf{y}_i\|_2 \leq \|\mathbf{v}_i\|_2$ and $\|\mathbf{y}_i\|_2^2 \leq \|\mathbf{v}_i\|_2^2 + \frac{1}{4} \sum_{j=1}^{i-1} \|\mathbf{v}_j\|_2^2$ for all $i \in [n]$.

2. (a) For all large enough $n \in \mathbb{Z}$, find an $n$-dimensional full-rank lattice in which the successive minima $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (in the $l_2$ norm) do not form a basis of the lattice. (Hint: Cesium Chloride)

(b) Show that for any 2-dimensional full-rank lattice $L$, the successive minima $\mathbf{v}_1, \mathbf{v}_2$ do form a basis of $L$. (Hint: Consider the lattice obtained by projecting $L$ on the one-dimensional subspace $\{\mathbf{v}_1\}^\perp$ and show that the projection of $\mathbf{v}_2$ must be a basis of this lattice)

(c) Let $L$ be a 2-dimensional lattice, and let $\mathbf{b}_1, \mathbf{b}_2$ be an $\delta$-LLL reduced basis with $\delta = 1$. Show that $\|\mathbf{b}_1\|_2 = \lambda_1(L)$.

(d) Among all 2-dimensional full-rank lattices with $\lambda_1(L) = 1$, which one has the smallest $\det(L)$? (This lattice is unique up to rotation).

3. Show that a $\delta$-LLL reduced basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of a lattice $L$ with $\delta = \frac{3}{4}$ satisfies the following properties.

(a) $\|\mathbf{b}_1\|_2 \leq 2^{(n-1)/4} (\det(L))^{1/n}$.

(b) For any $1 \leq i \leq n$, $\|\mathbf{b}_i\|_2 \leq 2^{(n-1)} \|\mathbf{b}_1\|_2$.

(c) For any $1 \leq i \leq n$, $\lambda_i(L) \geq 2^{-(n-1)/2} \|\mathbf{b}_i\|_2$.

(d) Let $\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*$ denote the associated dual basis. Show that $\|\mathbf{b}_n^*\|_2 \leq 2^{(n-1)/2} \lambda_1(L^*)$. (Hint: Use Exercise 1 from the last homework)

4. Show that the LLL analysis is essentially tight: for $1/4 < \delta \leq 1$, construct an $\delta$-LLL reduced basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ for some lattice $L$ such that $\|\mathbf{b}_1\|_2 \geq \sqrt{1 - 1/(4\delta)} (2/\sqrt{4\delta - 1})^{n-1} \lambda_1(L)$. 

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