The following assignments are due on Friday April 5 at 4:00 pm. If you need to turn in your homework electronically, prepare one pdf file containing it all. Even better, print everything and bring it to class or put it under the door of WWH612.

Consider finite element approximations of the homogeneous Dirichlet problem for Poisson’s equation in a polygonal bounded domain $\Omega$ in the plane,

$$-\Delta u(x, y) = f(x, y), \forall (x, y) \in \Omega, \quad u(x, y) = 0, \forall (x, y) \in \partial \Omega.$$ 

Partition $\Omega$ into triangular elements; the intersection of the closure of any pair of distinct elements of the triangulation is either empty, a full edge common to the two elements of a common vertex.

This problem can also be written in variational form

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx = f(v) := \int_{\Omega} f v dx.$$ 

Introduce nodes at all interior vertices of the triangulation and, when we consider the continuous piecewise quadratic elements $P_2$, also a node at the midpoint of each edge of the triangles. Note that on each element a function in $P_2$ is spanned by $\{1, x, y, x^2, xy, y^2\}$. The piecewise linear $P_1$ have nodes only at the triangle vertices and are spanned by the first three functions in the set just specified.

1. Characterize the standard nodal basis functions for $P_1$ and $P_2$. What are the supports of these functions, i.e., the sets where they do not vanish?

2. The stiffness matrix of the problem is generated by using the bilinear form $a(u, v)$ and the nodal basis functions of the finite element space. Characterize the sparsity of the resulting matrix for the $P_1$ and $P_2$ cases.

3. Each triangle will contribute a quadratic form to the global quadratic form defined by the stiffness matrix of the finite element problem. What is the size of the local quadratic forms if we work with $P_2$?
4. Show that the matrices of these local quadratic forms are symmetric, singular, and positive semi-definite.

5. Show that the stiffness matrix $P_1$ case will satisfy the maximum principle if all the triangles are equilateral.

6. Show that we do not have a maximum principle if we instead work with $P_2$.

7. Show that we do not have a maximum principle for $P_1$ if at least one of the triangles has an angle larger than $\pi/2$.

8. Briefly outline what we should do if we replace the Dirichlet condition by a Neumann condition
   \[ \frac{\partial u}{\partial n} = g_N \]
   on a subset $\partial \Omega_N$ of $\partial \Omega$. Will the stiffness matrix remain symmetric, positive definite?

9. What might be the advantage of using the more complicated $P_2$ finite elements instead of $P_1$? On the same triangulation, how much larger will the stiffness matrix for $P_2$ be than that of $P_1$? What about the rates of convergence in the two cases if the solution of the PDE is smooth? (Be specific on how you measure the error.)