1. Consider the simple one-dimensional model problem

\[- \frac{d^2 u}{dx^2} = f(x), \ 0 \leq x \leq 1, \ u(0) = u(1) = 0.\]

(a) Approximate it by using a three-point formula after introducing a uniform mesh with mesh size \(h\). First consider a right hand side with components \(f(x_i)\). Select a reasonably smooth function \(u(x)\), compute \(f(x)\), and show experimentally that you get second order convergence; find the solution for at least two mesh sizes.

When solving the linear system explore that the matrix will be symmetric, tridiagonal, and positive definite.

(b) Next modify the right hand side by replacing the components of the right hand side \(f(x_i)\) by an appropriate linear combination of \(f(x_{i-1}), f(x_i), \) and \(f(x_{i+1})\). Show theoretically and experimentally that we can obtain fourth order accuracy.

(c) Finally, write done the two leading terms of the truncation error for the more advanced difference approximation. Approximate these terms by using the approximate values obtained in the second set of experiments and try to approximately eliminate the effect of these leading terms of the truncation error by solving the finite difference equation with a modified right hand side.

Try to establish experimentally that we can obtain better than fourth order accuracy using this deferred correction method. Note that when approximating the new terms on the right hand side, we need to
approximate higher order derivatives. This can be done by computing interpolation polynomials using values at enough mesh points and computing the required derivatives. Note that at the mesh points close to the end points of the interval, there are mesh points “missing” on one side and that there we cannot use the same number of points on the left and right of the meshpoint.

2. Set up the five-point finite difference approximation of Poisson’s equation

\[-\Delta u = f, \text{ in } \Omega, \quad u(x, y) = 0, \quad (x,y) \in \partial\Omega,\]

where $\Omega$ is the unit square and $\partial\Omega$ its boundary. Use uniform meshes.

(a) Solve the resulting linear system of equations using sparse MATLAB. By varying the mesh size, and therefore the order of the linear system, estimate the growth of the number of nonzeros of the triangular factors.

Note that it is known that the nested dissection ordering, briefly discussed in class and described, e.g., in George’s SINUM volume 10(2) paper, will give an order of $n \log(n)$ such nonzeros. Compare your experimental findings with that result and also with the number of nonzeros that would be obtained if a band Cholesky method was used after ordering the unknowns row by row in the square domain.

(You could try to fit your data using linear least squares and a model such as $An \log n + Bn + C$, where $A$, $B$, and $C$ are constants.)

(b) Generate a right hand side $f$ of the Poisson problem from a smooth function $u(x, y)$, which vanishes on $\partial\Omega$. Verify numerically that the discretization error is of order $h^2$. Also try some $u(x, y)$ which is not very smooth.

(c) Change the domain to a triangle with vertices at $(-1/2, 0)$, $(1/2, 0)$, and $(0, 1)$. Introduce a mesh with mesh points at the vertices of the triangle and a uniform mesh size $h$ in the $x$– and $y$–directions. Again use the five point formula with an approximation at the mesh points next to the boundary given by the special formulas on pp. 155-156 of the text.

Will the approximation still be of second order in spite to the fact that the truncation error at points next to the boundary is only of first order?

Is the matrix still symmetric or can it be made symmetric by some simple scaling of some of the rows and columns of the matrix?

Try to answer the question on the accuracy by some careful experiments. When constructing functions $u(x, y)$ for your experiments and a right hand side $f(x, y) = -\Delta u(x, y)$, the easiest way to do so might be to use a product of three functions, each of which vanishes on one of the edges of the triangular domain.