Exercise 7.1. Assume that we wish to interpolate the function \( f(x) \) in the interval \([a, b]\) at \( n+1 \) equally spaced points \( \{x_i\}, i = 0, \ldots, n \), where \( x_i = a + ih \) with \( h = (b-a)/n \). If \( p_n \) is the interpolating polynomial, suppose that we wish to evaluate the error in \( p_n(x) \) at \( x = x_0 + th \), with \( 0 \leq t \leq n \). For some \( \xi \in [a, b] \),

\[
f(x) - p_n(x) = t(t-1)(t-2)\cdots(t-n)h^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!},
\]

and this error is closely related to the so-called “error factor polynomial”:

\[
\Pi_n(t) = t(t-1)(t-2)\cdots(t-n).
\]

(a) What are the roots of \( \Pi_n(t) \)?

(b) It can be shown that a maximum of \( |\Pi_n(t)| \) occurs when \( n - 1 < t < n \). Based on this property and the result of (a), what behavior of \( \Pi_n(t) \) might affect the accuracy of the interpolating polynomial at equally spaced points?

Exercise 7.2. Choose a value of \( n \geq 5 \) and make up your own two sets of data consisting of \( n+1 \) pairs \( \{x_i, f_i\}, i = 0, \ldots, n \), where the values \( \{x_i\} \) are distinct in each data set and \( f_i \) represents the value of an unknown function evaluated at \( x_i \). In at least one of your two data sets, the points \( \{x_i\} \) should not be equally spaced.

Interpolate the values of \( \{f_i\} \) in both sets of data with:

(i) the interpolating polynomial of degree at most \( n \);

(ii) the ‘not a knot’ spline computed by Matlab’s ‘spline’ command; and

(iii) the shape-preserving piecewise cubic computed by Matlab’s ‘pchip’ command.

For both data sets, calculate the values of the interpolating polynomial, the spline, and the ‘pchip’ interpolant at a reasonable number (say, 41) of equally spaced points in the interval defined by the points \( \{x_i\} \). Plot the interpolants both separately and superimposed (as in Figure 12 of Handout 9).

Choose your data sets so that “interesting” things happen. Comment on how the interpolated functions differ qualitatively. For both data sets, which choice of interpolant seems to you to provide the best “feeling” for the data? Explain why.

Exercise 7.3. Suppose that you are given the following tabulated data \( \{x_i, f_i\}, i = 0, \ldots, n \), where \( f_i = f(x_i) \) for some function \( f \):

<table>
<thead>
<tr>
<th>i</th>
<th>( x_i )</th>
<th>( f_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.52</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.68</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2.12</td>
</tr>
</tbody>
</table>
Assume that interval $i$ means $[x_{i-1}, x_i]$, and note that the values $\{x_i\}$ are not equally spaced. Consider the following piecewise cubic, where $s_i$ is the local cubic in interval $i$, and $u = x - x_{i-1}$ if $x$ is in interval $i$:

- Interval 1: $s_1(x) = 1 - 0.5u + 0.02u^3$
- Interval 2: $s_2(x) = 0.52 - 0.44u + 0.06u^2 + 0.1u^3$
- Interval 3: $s_3(x) = 0.68 + u + 0.66u^2 - 0.22u^3$

Is this the interpolating natural piecewise cubic spline for these data? Explain why or why not, and show your associated calculations. (Remember that, in the natural spline interpolant, $s''(x_0) = 0$ and $s''(x_n) = 0$.)

**Exercise 7.4.** In discussing the interpolating polynomial that matches a function $f(x)$ at $n + 1$ specified distinct points $x_0, \ldots, x_n$, it has always been stated that the interpolating polynomial is of degree “at most $n$”. Explain when the interpolating polynomial that matches a function $f(x)$ at three distinct points will have (i) degree 0 and (ii) degree 1. Illustrate these two cases with (simple) examples.

**Exercise 7.5.** Let $I(b)$ denote the exact integral

$$I(b) = \int_1^b \frac{dt}{t} = \ln b.$$  

For a given value of $b$, let $M(b)$ denote the estimate of $I(b)$ from the midpoint rule; $T(b)$ the estimate of $I(b)$ from the trapezoid rule; and $S(b)$ the estimate of $I(b)$ from Simpson’s rule. For $b = 1.5$ and $b = 2$, do the following:

(i) compute the exact integral $I(b)$;

(ii) compute $M(b)$ and the error $I(b) - M(b)$;

(iii) compute $T(b)$ and the error $I(b) - T(b)$;

(iv) compute $S(b)$ and the error $I(b) - S(b)$;

(v) comment on how well the actual errors correspond to the estimates derived in class, explaining why the error estimates are more accurate for the first value of $b$.

**Exercise 7.6.** Consider the integral (and its known exact value)

$$\int_0^\pi \sin^2 4t \, dt = \frac{\pi}{2}.$$  

Explain what would go wrong if you tried to approximate this integral with the midpoint rule, the trapezoid rule, or Simpson’s rule. How could you obtain an accurate approximation to this integral?