Exercise 7.1. Define $A$ and $b$ as

$$A = \begin{pmatrix} 6 & 3 \\ 4 & 1.99999 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 6 \\ 2 \end{pmatrix}.$$ 

(a) Give the singular values of $A$ and $\text{cond}(A)$.

(b) Compute $x_1$, the vector that minimizes $\|b - Ax_1\|_2^2$. This is, in effect, using a liberal rank-estimation strategy in which $A$ is treated as having rank 2 and the small singular value is treated as "significant". Give the vector $x_1$, $\|x_1\|_2$, the residual vector $r_1 = b - Ax_1$, and $\|r_1\|_2$.

(c) Now try a conservative strategy in which $A$ is deemed to have rank one and replaced by a related rank-one matrix $\hat{A}$ whose smallest singular value is zero. Give $\hat{A}$ and explain how you obtained it.

Give the matrix $B = A - \hat{A}$ and $\|B\|_2$. Give the solution $x_2$ of $\min \|b - \hat{A}x_2\|_2^2$ (this can be done using the Matlab ‘backslash’ command) and $\|x_2\|_2$. Give the residual $r_2 = b - \hat{A}x_2$ (be sure to use the original matrix $A$) and $\|r_2\|_2$.

Comment on the differences in the size of the solution and residual compared to those of part (b).

(d) Compute the $QR$ factorization of $A$. Explain how the ill-conditioning of $A$ is revealed in the upper-triangular factor $R$.

For Exercises 7.2, 7.3, and 7.4, you may find it convenient to use the built-in Matlab functions $\text{polyfit}$, $\text{polyval}$, and $\text{vander}$. If you use these built-in functions, keep in mind that the constant term in the polynomial is the last element in the vector of coefficients produced by $\text{polyfit}$ and (similarly) that, using the Matlab command $\text{vander}$, the column associated with the constant term (i.e., the column whose elements are all equal to one) is the last column of the Vandermonde matrix.

Exercise 7.2. Consider the function $f(x) = \sin(x)$ and suppose that we wish to approximate $f$ in the interval $[-1, 2]$ (where the units measure radians). Compute the three interpolating polynomials of degrees $n = 1, 2, 3$ at $n + 1$ equally spaced points in this interval, including the endpoints. List the coefficients of each polynomial. For each value of $n$, plot the values of $\sin$ and the corresponding interpolating polynomial at 20 equally spaced points in $[-1, 2]$, also designating the $n + 1$ interpolation points. Comment on whether the interpolating polynomials give a good fit to $\sin(x)$ in this interval.

Exercise 7.3$. Suppose that you are asked to use a 10th degree polynomial to interpolate census data between 1950 and 2000 by fitting data at the 11 interpolation points 1950, 1955, 1960, 1965, ..., 2000, using the monomial form of the interpolating polynomial. Invoke at least one principle of numerical computing (illustrated by a computation involving the Vandermonde matrix) to explain why this is undesirable.

Exercise 7.4. Consider the interval $[-1, 1]$ and Runge’s function:

$$\rho(x) = \frac{1}{1 + 25x^2}. \quad (1)$$

(a) For $n = 2, n = 11,$ and $n = 20,$ compute the interpolating polynomial of degree $n$ that matches $\rho$ at $n + 1$ equally spaced points in $[-1, 1]$ (including the endpoints). Print the interpolation points and the coefficients of the interpolating polynomial in monomial form.

For each $n,$ evaluate the interpolating polynomial at 41 equally spaced points in the interval $[-1, 1]$ and compute the error at each of these points, i.e., the value of $\rho(x) - p_n(x),$ where $p_n$ is the interpolating polynomial. Plot the values of the error and comment on your results, in particular on any differences in the size of the error in different regions of the interval.

(b) For $n = 2, n = 11,$ and $n = 20,$ compute the interpolating polynomial of degree $n$ that matches $f$ at the following $n + 1$ points:

$$z_k = \cos \left( \frac{(2k + 1)\pi}{2(n + 1)} \right), \quad k = 0, 1, \ldots, n. \quad (2)$$

These are the zeros of the Chebyshev polynomial $T_{n+1}(x).$ For example, when $n = 2,$ these points are $z_1 = \cos(\pi/6) \approx 0.86603,$ $z_2 = \cos(\pi/2) = 0,$ and $z_3 = \cos(5\pi/3) \approx -0.86603.$

For each $n,$ print the interpolation points $\{z_k\},$ $k = 0, \ldots, n,$ from (2) and the coefficients of the interpolating polynomial in monomial form that matches $\rho$ at $\{z_k\}.$

Then evaluate the interpolating polynomial at 41 equally spaced points in the interval $[-1, 1],$ and compute $\rho(x) - p_n(x)$ at those points. Plot the error and comment on your results.

(c) Comment on any notable differences between the results of (a) and (b) in terms of the accuracy of the interpolating polynomial at points other than the interpolation points.

**Exercise 7.5.** This problem involves determination of a linear polynomial $ax + b$ that gives the “best fit” to the function $x^3$ in the interval $[-1, 1],$ using the $\ell_2$ norm. Let

$$E_2^2(a, b) = \int_{-1}^{+1} (x^3 - (ax + b))^2 \, dx$$

What are the values of $a$ and $b$ for which $E_2^2(a, b)$ is minimized? Show how you obtained your answer. What is the value of $E_2^2(a, b)$ for these coefficients? For a reasonable number of equally spaced points in $[-1, 1],$ make a plot showing both $x^3$ and $ax + b$ (using the optimal $a$ and $b$). Where in the interval does the maximum error occur?