In all problems involving vector or matrix norms, assume that the two-norm is used unless stated otherwise. When equations involving matrices and vectors are given, assume that the dimensions of the matrices and vectors are compatible.

Exercise 4.1. Let \( \{ \beta_i \} \), \( i = 1, \ldots, n \), be a specified set of \( n \) nonnegative numbers, \( n > 1 \), ordered so that

\[
\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0.
\]

We wish to find a set of \( n \) nonnegative numbers \( \{ \alpha_i \} \), \( i = 1, \ldots, n \), that satisfies certain properties, where at least one \( \alpha_i \) is strictly positive.

(a) Let \( \rho \) denote the ratio

\[
\rho = \frac{\alpha_1 \beta_1 + \cdots + \alpha_n \beta_n}{\alpha_1 + \cdots + \alpha_n}.
\]

Prove that \( \rho \) is maximized when \( \alpha_1 > 0 \) and \( \alpha_2 = \cdots = \alpha_n = 0 \). Under what conditions on the values of \( \{ \beta_i \} \) is the maximum value of \( \rho \) achieved for other choices of \( \{ \alpha_i \} \)?

(b) For what choices of \( \{ \alpha_i \} \) is the value of \( \rho \) minimized? Justify your answer.

Exercise 4.2.

(a) Show that, if the square matrices \( B \) and \( C \) are nonsingular, then \( \text{cond}(BC) \leq \text{cond}(B) \text{cond}(C) \), where the condition number is measured in any norm.

(b) Show that \( \text{cond}(A^T) = \text{cond}(A) \), measured in the matrix two-norm. Does this result hold for the matrix one- and infinity-norms? Explain your answer, giving examples if your answer is “no” in either case.

Exercise 4.3. Given a nonsingular matrix \( A \) and a vector \( b \), let \( x \) denote the exact solution of \( Ax = b \).

(a) If \( \|b\| = 10^{-6} \) and \( \|x\| = 1 \), explain what, if anything, can be deduced about the smallest singular value of \( A \)?

(b) If the conditions of part (a) apply and \( \|A\| = 1 \), what do we know about \( \text{cond}(A) \)?

(c) If \( \|A\| = 1 \), \( \|b\| = 1 \) and \( \|x\| = 1 \), can we conclude that \( A \) is well conditioned? Explain why or why not.

Exercise 4.4. Let \( A \) be nonsingular and \( X \) be an approximation to \( A^{-1} \). Consider the residual matrix \( R \) defined by \( R = I - AX \).

(a) Show that

\[
\frac{\|A^{-1} - X\|}{\|A^{-1}\|} \leq \|R\|,
\]

where \( \|\cdot\| \) denotes any subordinate norm. This relation shows that, when computing the inverse matrix, a small residual matrix guarantees that \( X \) is close (in norm) to \( A^{-1} \).
(b) Show that
\[
\| I - X \| \leq \text{cond}(A) \| I - AX \|. \tag{1}
\]
A conclusion from this relation is that, if \( A \) is ill-conditioned, there could be a matrix \( X \) that is a good approximate right inverse of \( A \) (in the sense that \( \| I - AX \| \) is small), but a poor approximate left inverse.

(c) For a given nonsingular matrix \( A \), is there always a matrix \( X \) for which equality holds in (1)? Explain why or why not.

(d) Devise your own \( n \times n \) ill-conditioned matrix \( A \) \((n > 1)\) and an approximate inverse \( X \) for which \( \| I - AX \| \) is small, but (1) holds with near-equality. Explain how you chose \( A \).

Exercise 4.5. Consider the matrix \( A \) and vector \( b \),
\[
A = \begin{pmatrix} 0.671 & -0.273 \\ -0.335 & 0.136 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 0.398 \\ -0.199 \end{pmatrix},
\tag{2}
\]
and let \( x \) be the exact solution of \( Ax = b \).

(a) Give the singular value decomposition of \( A \). What is \( \text{cond}(A) \)?

(b) Give a small perturbation \( \delta_1 \) such that the solution \( x_1 \) of \( Ax_1 = b + \delta_1 \) satisfies
\[
\frac{\| x - x_1 \|}{\| x \|} \approx \text{cond}(A) \frac{\| \delta_1 \|}{\| b \|}. \tag{3}
\]
Include the numerical values of \( \delta_1, \| \delta_1 \|, x_1, \| x - x_1 \|, \) the ratios \( \| x - x_1 \| / \| x \| \) and \( \| \delta_1 \| / \| b \| \), and the right-hand side of (3). Explain how you chose \( \delta_1 \), referring to the SVD.

(c) Give a small perturbation \( \delta_2 \) such that the solution \( x_2 \) of \( Ax_2 = b + \delta_2 \) satisfies
\[
\frac{\| x - x_2 \|}{\| x \|} \ll \text{cond}(A) \frac{\| \delta_2 \|}{\| b \|},
\]
where \( \ll \) means “is much less than”. Please give \( \delta_2, \| \delta_2 \|, x_2, \| x - x_2 \|, \) and the ratios \( \| x - x_2 \| / \| x \| \) and \( \| \delta_2 \| / \| b \| \). Explain how you chose \( \delta_2 \), referring to the SVD.

(d) Choose two small “random” perturbations \( \delta_i, i = 3 \) and \( i = 4 \). Compute the solutions of the associated linear systems \( Ax_i = b + \delta_i \) and the ratios \( \| x - x_i \| / \| x \| \) and \( \| \delta_i \| / \| b \| \). Comment on the results, in particular on any resemblance (or lack of resemblance) to the results of (b) and (c).

(e) Let \( b_a = (-45, 22)^T \). Compute the solution of \( Ax = b_a \) using Matlab’s backslash operation. Devise your own perturbation matrix \( \delta A \) that is small in norm such that the solution \( \tilde{x}_A \) of
\[
(A + \delta A) \tilde{x}_A = b_a
\]
satisfies
\[
\frac{\| x - \tilde{x}_A \|}{\| x \|} \approx \text{cond}(A) \frac{\| \delta A \|}{\| A \|}.
\]
Please give \( \delta A, \| \delta A \|, \| x - \tilde{x}_A \|, \) and the ratios \( \| x - \tilde{x}_A \| / \| x \| \) and \( \| \delta A \| / \| A \| \). Explain how you chose \( \delta A \).

Exercise 4.6. Let \( A \) be a nonsingular \( n \times n \) matrix with smallest singular value \( \sigma_n > 0 \).

(a) Let \( B \) be any \( n \times n \) singular matrix. Show that \( \| A - B \|_2 \geq \sigma_n \).
(b) Explain how to use the SVD of $A$ to construct a singular matrix $B$ such that $\|A - B\| = \sigma_n$, where $\|\cdot\|$ is the two-norm. Give a mathematical form for the resulting $B$. Is $B$ necessarily unique? Explain why or why not.

(c) Given the matrix $A$ from (2), use the technique from part (b) of this problem to compute a singular matrix $B$ for which $\|B - A\| = \sigma_n$, showing at least 6 figures for each entry of $B$. Compute the SVD of $B$, give its computed singular values, and give the computed value of $\|A - B\|$. If the smallest computed singular value of $B$ is not exactly zero, comment on why this might be the case.

Exercise 4.7.

(a) Consider an $n \times n$ upper-triangular matrix $U$ such that $u_{11}u_{22}\cdots u_{n-1,n-1} \neq 0$, but $u_{nn} = 0$, i.e., $U$ is singular. Give a general algorithm for computing a nonzero vector $x$ such that $Ux = 0$. Verify your algorithm by applying it to the matrix

$$U = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & \end{pmatrix}.$$

What is the general form of $x$ satisfying $Ux = 0$ for this particular $U$?

(b) Suppose that $R$ is an $n \times n$ upper-triangular matrix and that $k$ of its diagonal elements are zero, where $n > k > 0$. We know that $R$ is singular, but is the rank of $R$ necessarily $n - k$? If “yes”, show why. If “no”, give an example that shows why not.

Exercise 4.8.

(a) Show that the product of two square upper-triangular matrices is upper-triangular.

(b) Given a nonsingular upper-triangular matrix $U$ whose diagonal elements are $\{u_{ii}\}$, show that (i) its inverse $U^{-1}$ is also upper triangular and (ii) the diagonal elements of $U^{-1}$ are the reciprocals of the diagonal elements of $U$.

(c) Using the result of (b) (and recalling that the matrix infinity norm is the maximum absolute row sum), show that

$$\|U\|_\infty \geq \max_i |u_{ii}| \quad \text{and} \quad \|U^{-1}\|_\infty \geq \frac{1}{\min_i |u_{ii}|}.$$  

These two inequalities imply that, measured in the infinity norm,

$$\text{cond}(U) \geq \frac{\max_i |u_{ii}|}{\min_i |u_{ii}|}. \quad (4)$$  

In practice, the lower bound on the right-hand side is often used as an estimate of the condition of an upper-triangular matrix.

Exercise 4.9. Consider the $n \times n$ lower-triangular matrix $L_n$ of the form

$$L_n = \begin{pmatrix} 1 \\ -1 & 1 \\ -1 & -1 & \ddots \\ \vdots & \iddots & 1 \\ -1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$
(a) What is the value of $\|L_n\|_\infty$? (Note: the infinity-norm.)

(b) Find a vector $x$ of unit infinity-norm such that $\|L_n x\|_\infty = \|L_n\|_\infty$.

(c) Consider the $n$-vector $y$ whose $i$th component is $(\frac{1}{2})^{n-i}$, i.e.,

$$y = \begin{pmatrix} 1 \\ \frac{1}{2^{n-1}} \\ \frac{1}{2^{n-2}} \\ \vdots \\ \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}.$$ 

What is the value of $\|y\|_\infty$? Display the vector $L_n y$. What is $\|L_n y\|_\infty$?

(d) Recall the result that, for any nonsingular matrix $A$, the relation $Ax = b$ implies that

$$\|A\| \geq \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \|A^{-1}\| \geq \frac{\|x\|}{\|b\|},$$

for any vector norm and its induced matrix norm. With this in mind, what do the results of (b) and (c) tell you about $\text{cond}(L_n)$ measured in the infinity norm?

(e) Let $U_n = L_n^T$ denote the transpose of $L_n$. If Gaussian elimination with partial pivoting is applied to $U_n$, what are the $LU$ factors? What is the associated lower bound on $\text{cond}(U_n)$ from the right-hand side of (4)? Is the lower bound a good estimate of the condition in this case?