Exercise 1.1. (Sequences and rates of convergence.)

(a) Compute and print the first 11 elements of the sequence \( \{x_k\} \), where \( x_k = c^{2^k} \) with \( c = .95 \), starting with \( x_0 \). Assuming that \( c \) is any constant satisfying \( 0 < c < 1 \), what is the limit of this sequence? Determine the order of convergence and the asymptotic error constant for this sequence.

(b) Compute and print out the first twelve elements of the sequence \( \{x_k\} \), where \( x_k = 1/k^k \), starting with \( k = 1 \). Determine the limit of the sequence. Is the sequence converging superlinearly or not? Explain.

Exercise 1.2. (Bisection–I.)

(a) Write a program that performs no more than a specified number \( \text{maxit} \) of bisection steps, given \( a_0 \) and \( b_0 \) (the endpoints of an initial interval \([a_0, b_0]\) with \( a_0 < b_0 \)), and a nonlinear function \( f(\cdot) \), where we are seeking a point \( x^* \) in \([a_0, b_0]\) such that \( f(x^*) = 0 \). At each bisection step, print the iteration \( k \), the endpoints of the \( k \)th interval of uncertainty \([a_k, b_k]\), and the values \( f(a_k) \) and \( f(b_k) \).

(b) Consider the function \( \hat{f}(x) = x^3 - 2.5x - 4.011 \). Show by hand calculation that a zero \( x^* \) of \( \hat{f}(x) \) must exist in \([-2, 4]\). What is the exact value of \( x^* \)? Do other real zeros exist? Why or why not?

(i) Run your bisection program on \( \hat{f} \) with \( a_0 = 0.5, b_0 = 3.1 \), and \( \text{maxit} = 12 \). Does the final interval of uncertainty contain \( x^* \)? Explain whether the results are what you expected.

(ii) Again using \( \hat{f}(x) \), run your bisection program with \( a_0 = -2, b_0 = 3.1 \), and \( \text{maxit} = 12 \). Does the final interval of uncertainty contain \( x^* \)? Explain whether the results are what you expected.

Exercise 1.3. (Bisection–II.)

Consider the function
\[
\tilde{f}(x) = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1.
\]
(Note that \( \tilde{f}(x) \equiv (x - 1)^7 \).) Let \( a_0 = 0.95, b_0 = 1.01 \), and \( \text{maxit} = 12 \).
(a) Show mathematically that \( \tilde{f} \) has only one zero, at \( x^* = 1 \).

(b) Run your bisection program, evaluating \( \tilde{f} \) using the expression in (1.1). What is the interval of uncertainty at the last bisection iteration? Explain whether or not the results are what you would expect from bisection and, if not, why.

(c) Run your bisection program a second time, this time applied to \( \bar{f}(x) = (x-1)^7 \), evaluated in this form, using the same \( a, b \), and \texttt{maxit} as in part (b). Given that, mathematically, \( \bar{f}(x) \equiv \tilde{f}(x) \), comment on your results, particularly any differences between the results of (b) and (c).

Exercise 1.4. (Newton.)

(a) Write a program that performs a sequence of no more than \texttt{maxit} Newton iterations for scalar zero-finding. The inputs to your program should be: (1) a subroutine that evaluates a differentiable function \( f(x) \) and its derivative \( f'(x) \), and (2) a starting point \( x_0 \). Your program should generate the Newton sequence \( \{x_k\} \) and terminate in one of two ways: after \texttt{maxit} Newton iterations, or when \( |f(x_k)| \leq 10^{-15} \). At each iteration, print the iteration number \( k \), \( x_k \), and \( f(x_k) \).

(b) Run your Newton program on the function \( \hat{f}(x) = x^3 - 2.5x - 4.011 \) from Exercise 1.2(b) with \( x_0 = 3 \) and \texttt{maxit} = 12. Explain whether the computed results are what you expect, especially the rate of convergence.

(c) Apply your Newton program to the function \( \tilde{f}(x) = (x-1)^7 \) with \( x_0 = 2.1 \) and \texttt{maxit} = 15. Comment on whether the computed results are what you expect, including the rate of convergence. Please explain the major qualitative difference in the behavior of the iterates from the results in part (b) of this exercise.

Exercise 1.5. (Secant.)

(a) Write a program that performs a sequence of no more than \texttt{maxit} secant iterations given the following inputs: a differentiable function \( f(x) \) and two initial points \( x_0 \) and \( x_1 \). Your program should generate the sequence \( \{x_k\} \) of secant iterates and terminate in one of two ways: after \texttt{maxit} secant iterations, or when \( |f(x_k)| \leq 10^{-15} \). At each iteration, print the iteration number \( k \), \( x_k \) and \( f(x_k) \).

(b) Run your secant program on the function \( \hat{f}(x) \) from Exercise 1.2(b), \( \hat{f}(x) = x^3 - 2.5x - 4.011 \), with \( x_0 = 0.5 \), \( x_1 = 3.1 \), and \texttt{maxit} = 15. Comment on whether the computed results are what you expect, including the behavior of the iterates and the rate of convergence.

(c) Apply your secant program again to \( \hat{f} \), but this time use \( x_0 = 3.1 \), \( x_1 = -2 \), and \texttt{maxit} = 25. Comment on whether the computed results are what you expect. Also comment on what can be concluded from comparing these results with those of bisection obtained in 1.2(b)(ii).