Exercise 1.1. Consider a hyperplane $a^T x = \beta$, where $x \in \mathbb{R}^n$, $a \neq 0$, and $\beta > 0$. Find the vector $x^*$ of smallest two-norm that lies on the hyperplane, i.e. such that $\|x^*\|_2 \leq \|x\|_2$ among all $x$ satisfying $a^T x = \beta$. What is $\|x^*\|_2$?

Exercise 1.2. Let $\mathcal{F}$ denote the set of points satisfying

$$\mathcal{F} = \{ x \in \mathbb{R}^n : A x = b \},$$

where $A$ is a nonzero $m \times n$ matrix of rank $r$, with $r < m$. Assume that $\mathcal{F}$ is not empty. Let $A_r$ denote a matrix composed of $r$ linearly independent rows of $A$ and let $b_r$ denote the $r$-vector of corresponding components of $b$. Show that $\mathcal{F}$ is identical to the set

$$\mathcal{F}_r = \{ x \in \mathbb{R}^n : A_r x = b_r \}.$$

Exercise 1.3. Consider minimizing $c^T x$ subject to $Ax = b$, where

$$A = \begin{pmatrix} -3 & 6 & 8 & -1 & -8 \\ -6 & 2 & 0 & 2 & -7 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 2 \\ -2 \end{pmatrix}.$$

(i) Find any feasible point $\bar{x}$. Is $\bar{x}$ optimal? Why or why not?

(ii) Find the unique vectors $c_N \in \text{null}(A)$ and $c_R \in \text{range}(A^T)$ such that $c = c_R + c_N$. Using these vectors, find a direction $p$ such that $Ap = 0$ and $c^T (\bar{x} + \alpha p) < c^T \bar{x}$ for all positive $\alpha$. 
Exercise 1.4. Suppose that exactly four foods are available to you: milk, chocolate chip cookies, chicken soup, and Brussels sprouts. Your daily diet today consists of 2 pints of milk, 24 cookies, 3 cups of soup and \( \frac{1}{2} \) pound of Brussels sprouts. (You actually despise Brussels sprouts, but include them because they are “good for you”.) You have recently been informed that you must reduce your daily intake of fat.

You wish to find a new diet consisting of nonnegative quantities of these four foods that maximizes enjoyment while satisfying two conditions: (1) your daily fat intake must be reduced by at least 3000 units compared to the amount consumed now; (2) you must consume at least 600 units of vitamin X, 300 units of vitamin Y, and 550 units of vitamin Z each day.

The relevant measurements for the four foods are shown, where the numerical entries denote the units shown:

<table>
<thead>
<tr>
<th>Food</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Fat</th>
<th>Enjoyment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milk (pint)</td>
<td>50</td>
<td>10</td>
<td>150</td>
<td>800</td>
<td>200</td>
</tr>
<tr>
<td>Cookies (dozen)</td>
<td>3</td>
<td>10</td>
<td>35</td>
<td>6000</td>
<td>6000</td>
</tr>
<tr>
<td>Soup (cup)</td>
<td>150</td>
<td>75</td>
<td>75</td>
<td>1000</td>
<td>3000</td>
</tr>
<tr>
<td>Sprouts (pound)</td>
<td>100</td>
<td>100</td>
<td>5</td>
<td>400</td>
<td>−200</td>
</tr>
</tbody>
</table>

(i) Formulate a linear program of the form “maximize \( c^T x \) subject to \( Ax \geq b \)” whose solution will tell you the best new diet, i.e. define \( A \), \( b \), and \( c \). Note that we want to maximize and that the variables must be nonnegative.

(ii) Find any vertex and compute the associated value of the objective function.

Exercise 1.5. Let \( A \) be a nonzero \( m \times n \) matrix, where \( m > 0 \) and \( n > 0 \). Assume that \( a \) is an \( n \)-vector that is linearly independent of the rows of \( A \). Let \( e_{m+1} \) denote the \((m + 1)\)th coordinate vector, and let \( \bar{A} \) denote the \((m + 1) \times n \) matrix

\[
\bar{A} = \begin{pmatrix} \ A \\ a^T \end{pmatrix}.
\]

Show that the equations \( \bar{A} p = e_{m+1} \) must be compatible.

Exercise 1.6. (Two easy parts of Farkas’ Lemma, not shown in class.) Let \( M \) be a nonzero \( k \times n \) matrix and \( c \) an \( n \)-vector.

(a) If \( c = M^T \lambda \) for some \( \lambda \geq 0 \), show that \( c^T p \geq 0 \) for all \( p \) such that \( Mp \geq 0 \).

(b) If \( c \neq M^T \lambda \) for any \( \lambda \), show that there exists \( p \) such that \( c^T p < 0 \) and \( Mp \geq 0 \).
Exercise 1.7. The linear program

minimize $-5x_1 - 6x_2 - 9x_3 - 8x_4$
subject to
$x_1 + 3x_2 + 4x_3 + 5x_4 \geq 7$
$-x_1 - 2x_2 - 3x_3 - x_4 \geq -5$
$-x_1 - x_2 - 2x_3 - 3x_4 \geq -3$
x_1 \geq 0
x_2 \geq 0
x_3 \geq 0
x_4 \geq 0

has a minimizer $x^* = (1, 2, 0, 0)^T$. (Trust me on this.)

(i) Find the active set at $x^*$. How many constraints are active?

(ii) Because $x^*$ is optimal, we know from Farkas’ Lemma that there must exist a nonnegative $\lambda_a$ such that $A_a^T \lambda_a = c$, where $A_a$ is the active-constraint matrix at $x^*$. Find two distinct nonnegative vectors $\lambda_a$ satisfying $A_a^T \lambda_a = c$.

Exercise 1.8. Consider the set of six inequality constraints $Ax \geq b$, with

$$A = \begin{pmatrix}
1 & 1 \\
0 & 1 \\
1 & 3 \\
0 & -1 \\
1 & 0 \\
-6 & 1
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}
4 \\
3 \\
6 \\
-6 \\
-1 \\
-18
\end{pmatrix}.$$

(a) Consider the point $\bar{x} = (5, 2)^T$ and direction $p = (-1, 1)^T$. Find the step $\gamma_i$ from $\bar{x}$ along $p$ that “hits” each constraint $a_i^T x \geq \beta_i$. For each constraint, determine whether $\bar{x}$ is feasible and state whether a positive step along $p$ causes the constraint to increase, decrease, or remain unchanged.

(b) Consider the point $\bar{x} = (2, 4)^T$ and direction $p = (-1, 1)^T$. Is $\bar{x}$ feasible? Determine the constraints active at $\bar{x}$ and compute the maximum positive feasible step that can be taken along $p$.

(c) Repeat part (b) for the point $\bar{x} = (2, 3)^T$ and $p = (-1, 1)^T$, and comment on any differences.

Exercise 1.9. Assume that the relevant linear constraints are $Ax \geq b$ for some generic $A$ and $b$. 
(a) Consider a point $\bar{x}$ where the active-constraint matrix is

$$A_a = \begin{pmatrix} -1 & -8 & 4 & -7 & -3 \\ -4 & 2 & 0 & 4 & -2 \\ -6 & 0 & -3 & 3 & 5 \\ 0 & 0 & 5 & 2 & 7 \end{pmatrix}.$$  

Is $\bar{x}$ a vertex? Explain why or why not. If so, is $\bar{x}$ degenerate or nondegenerate? Using Matlab (or any appropriate language of your choice), try to compute a direction $p$ such that a positive move along $p$ from $\bar{x}$ will increase the residual of the third constraint, but will leave the other residuals unchanged. Explain why, for the given $A_a$, such a $p$ must exist. Is $p$ unique? Explain. Using the row ordering of $A_a$ as given above, what will be the indices of the active constraints after a positive step from $\bar{x}$ along $p$?

(b) Repeat part (a), assuming that $\bar{x}$ is a point with the following active-constraint matrix:

$$A_a = \begin{pmatrix} 0 & 3 & 6 & -5 & -4 \\ 9 & 1 & 6 & -4 & 5 \\ 0 & 1 & 6 & -5 & -4 \\ -3 & -4 & -5 & 3 & 2 \\ 3 & 5 & -6 & 3 & 2 \\ -8 & -3 & 0 & 7 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Is $\bar{x}$ a vertex? Explain why or why not. If so, is $\bar{x}$ degenerate or nondegenerate? Comment on your results.

**Exercise 1.10.** Let $A$ be an $m \times n$ matrix.

(a) Show that if $b = (\beta_1, \ldots, \beta_m)^T$ is an $m$-vector such that $\beta_i \leq 0$ for $i = 1, \ldots, m$, then at least one feasible point must exist for the combined constraints

$$Ax \geq b, \quad x \geq 0.$$  

Is the result true for a general vector $b$? Explain why or why not.

(b) Consider the constraints $Ax \geq b$ and $x \geq 0$ for a general vector $b$, and assume that a feasible point exists. Must a vertex exist? Explain why or why not.

**Exercise 1.11.** Consider the two constraints $x_1 - x_2 \geq 0$ and $x_1 + 2x_2 \leq 6$, which intersect at $\bar{x} = (2, 2)^T$. Suppose that we wish to add a third constraint, $\alpha x_1 + x_2 \geq \gamma$, with $\alpha \geq 0$. 


(a) Given $\alpha$, what must be the value of $\gamma$ to ensure that the new constraint intersects the first two constraints at $\bar{x}$?

(b) With $\gamma$ taken as the value determined in part (a), analyze the role of $\alpha$ in the existence or non-existence of feasible directions at $\bar{x}$ with respect to all three constraints. Can you find values $\alpha_1 \geq 0$ and $\alpha_2 \geq \alpha_1$ such that feasible directions with respect to all three constraints exist if $\alpha_1 \leq \alpha \leq \alpha_2$, but no such feasible directions exist if $0 \leq \alpha < \alpha_1$ or if $\alpha > \alpha_2$? Explain your answer.

(c) Draw a figure that shows the geometric significance of the answers to part (b).

**Exercise 1.12.** Let $c$ be a nonzero $n$-vector. Using only the properties of an optimal solution of a linear program, find the solution $x^*$ of

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad l \leq x \leq u,
\end{align*}$$

where $l$ and $u$ are $n$-vectors with finite components that satisfy $l_i \leq u_i$ for $i = 1, \ldots, n$. 